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Weighted irregular Gabor tight frames and dual systems using windows in the Schwartz class

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Abstract

We give a characterization for the weighted irregular Gabor tight frames or dual systems in $L^2(\mathbb{R}^n)$ in terms of the distributional symplectic Fourier transform of a positive Borel measure on $\mathbb{R}^{2n}$ naturally associated with the system and the short-time Fourier transform of the windows in the case where the window (or at least one of the windows in the case of dual systems) belongs to $\mathcal{S}(\mathbb{R}^n)$. This result implies that, for certain classes of windows such as generalized Gaussians or "extreme-value" windows, the only weighted irregular Gabor tight frames (or even dual systems with both windows in the same class) that can be constructed with these windows are the trivial ones, corresponding to the measure $\mu = 1$ on $\mathbb{R}^{2n}$. Furthermore, we show that, if a such Gabor system admits a dual which is of Gabor type, then the Beurling density of the associated measure exists and is equal to one.

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1. Introduction

Let $g \in L^2(\mathbb{R}^n)$ be a window function with $\|g\|_2 = 1$. The corresponding *short-time Fourier transform* is the mapping $\mathcal{V}_g : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ defined, for $f \in L^2(\mathbb{R}^n)$, by

$$\mathcal{V}_g f(x, \nu) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \nu \cdot t} \overline{g(t - x)}\, dt, \quad (x, \nu) \in \mathbb{R}^{2n}.$$

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(Note that in the following, $\mathbb{R}^{2n}$ will be identified with $\mathbb{R}^n \times \mathbb{R}^n$, and thus, the notation $(x, \nu) \in \mathbb{R}^{2n}$ means, in particular, that both x and ν belong to $\mathbb{R}^n$.) The following result is central in the theory of the short-time Fourier transform (see [9, Corollary 3.2.2]).

Theorem 1.1. *Given $g \in L^2(\mathbb{R}^n)$ with $\|g\|_2 = 1$, we have*

$$\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(x,\nu)\right|^2 dx\, d\nu = \int_{\mathbb{R}^n} \left|f(t)\right|^2 dt, \quad f \in L^2\left(\mathbb{R}^n\right). \tag{1.1}$$

The short-time Fourier transform is thus, for a fixed g as above, an isometric linear transformation from $L^2(\mathbb{R}^n)$ onto a closed proper subspace of $L^2(\mathbb{R}^{2n})$ consisting of continuous functions. The identity (1.1) immediately yields an integral representation formula for functions in $L^2(\mathbb{R}^n)$ in terms of the continuous Gabor system $\{e^{2\pi i\nu\cdot t} g(t-x)\}_{(x,\nu)\in\mathbb{R}^{2n}}$:

$$f(t) = \int_{\mathbb{R}^{2n}} \mathcal{V}_g f(x,\nu) e^{2\pi i \nu\cdot t} g(t-x)\, dx\, d\nu, \quad f \in L^2\left(\mathbb{R}^n\right). \tag{1.2}$$

This formula represents f as a continuous superposition of elementary signals given by modulations and translations of the window function g. One major goal of Gabor analysis is to obtain discrete representation for functions in $L^2(\mathbb{R}^n)$ analogous to the one given in (1.2). In fact, most of the work done in Gabor analysis concerns systems of the form $\{e^{2\pi i b l\cdot t} g(t-ak)\}_{(k,l)\in\mathbb{Z}^n\times\mathbb{Z}^n}$, where $a, b > 0$ are two parameters. More recently, several authors have begun to investigate irregular Gabor systems $\{e^{2\pi i\nu\cdot t} g(t-x)\}_{(x,\nu)\in\Lambda}$ where Λ is a discrete set in the time–frequency space [1–5,7,18,19,21,27–29] as well as weighted irregular Gabor systems, i.e. systems of the form $\{w(x,\nu)^{1/2} e^{2\pi i\nu\cdot t} g(t-x)\}_{(x,\nu)\in\Lambda}$ where Λ is a discrete set in the time–frequency space and w is a positive function defined on Λ [16]. (The term "irregular" in this context might be somewhat misleading as it also includes the "regular" case.) If such a system forms a Parseval tight frame for $L^2(\mathbb{R}^n)$, we have, by definition,

$$\sum_{(x,\nu)\in\Lambda} w(x,\nu) \left|\mathcal{V}_g f(x,\nu)\right|^2 = \int_{\mathbb{R}^n} \left|f(t)\right|^2 dt, \quad f \in L^2\left(\mathbb{R}^n\right). \tag{1.3}$$

Introducing the positive measure μ on $\mathbb{R}^{2n}$ defined by

$$\mu = \sum_{(x,\nu)\in\Lambda} w(x,\nu)\delta_{(x,\nu)},$$

where $\delta_{(x,\nu)}$ denotes the Dirac mass at the point (x,ν), we can rewrite Eq. (1.3) as

$$\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(x,\nu)\right|^2 d\mu(x,\nu) = \int_{\mathbb{R}^n} \left|f(t)\right|^2 dt, \quad f \in L^2\left(\mathbb{R}^n\right). \tag{1.4}$$

The analogue of the reconstruction formula (1.2) now reads

$$f(t) = \int_{\mathbb{R}^{2n}} \mathcal{V}_g f(x,\nu) e^{2\pi i\nu\cdot t} g(t-x)\, d\mu(x,\nu), \quad f \in L^2\left(\mathbb{R}^n\right). \tag{1.5}$$

The discrete and continuous Gabor expansions considered above lead naturally to the following general question: given a window $g \in L^2(\mathbb{R}^n)$, which positive Borel measure μ on $\mathbb{R}^{2n}$ satisfy the identity (1.4)? It is also clear that the problem of constructing discrete, weighted irregular Gabor tight frames with a given window g is equivalent to finding discrete measures μ which satisfy the identity (1.4). Another question that we will address is the following. We know that, if g has norm 1, the Lebesgue measure on $\mathbb{R}^{2n}$, which we identify with the constant function 1, is always a solution of our problem (1.4). Are there windows $g \in L^2(\mathbb{R})$ for which this positive measure is the *only* solution of our problem? In such cases, it is clear that no (discrete) weighted irregular Gabor system constructed with the given windows will yield a tight frame for $L^2(\mathbb{R}^n)$. For technical reasons, we will restrict our analysis to windows belonging to the Schwartz class (or, with at least one of the windows in the Schwartz class in the case of dual systems). Indeed, these windows are well suited for Gabor analysis as they are very well localized in the time–frequency space. More importantly, it is not even clear how to state our main results in general, if this assumption is not made, as they would involve products of distributions which are not well defined.

The paper is organized as follows. One of our main goals will be to obtain a characterization of the positive measures μ on $\mathbb{R}^{2n}$ which satisfy the identity (1.4) in the case where the window function g belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This characterization involves the symplectic Fourier transform of the measure and, in order to make sense of it, we first need to show that any positive measure μ satisfying the identity (1.4) for a given window g must be a tempered measure. In fact, we show in Section 2 that if a window $g \in L^2(\mathbb{R})$ satisfies the Bessel condition with respect to the measure μ, which means that there exists a constant $C_1 \geqslant 0$, such that

$$\int\limits_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(x,\nu)\right|^2 d\mu(x,\nu) \leqslant C_1 \int\limits_{\mathbb{R}^n} \left|f(t)\right|^2 dt, \quad f \in L^2\left(\mathbb{R}^n\right), \tag{1.6}$$

then the measure μ must be *translation-bounded* and thus tempered (Proposition 2.2). Conversely, we also show that the Bessel condition (1.6) always holds if μ is translation-bounded and the window g belongs to the Schwartz class (Proposition 2.4).

In Section 3, we state our characterization of the identity (1.4) (Theorem 3.2) and use it to show that the uniqueness of the measure μ appearing in (1.4) for a fixed window g is equivalent to the non-vanishing of the short-time Fourier transform $\mathcal{V}_g g$ (Corollary 3.3). We show, in particular that this last property is satisfied by the class of *generalized Gaussians* windows (Theorem 3.10) and also by the so-called "*extreme-value*" windows (Proposition 3.11). This implies, of course, that no discrete, weighted irregular Gabor system can yield a tight frame for $L^2(\mathbb{R}^n)$ for such windows. In the case of *even* windows, we prove, using a result of Hudson [14], that the generalized Gaussians are the only ones with the property that $\mathcal{V}_g g$ does not vanish and thus for which the uniqueness of the measure in (1.4) holds (Proposition 3.8). In the case of odd windows, non-uniqueness always occurs (Proposition 3.9). We also show that, if the window is a generalized Gaussian multiplied by a polynomial, then, although there might be more than one measure μ satisfying (1.4), none of these can be discrete, thus preventing again the existence of discrete irregular Gabor tight frame constructed with such windows (Theorem 3.13).

In Section 4, we consider the problem of characterizing the "*duals of Gabor type*" associated with a window g in the Schwartz class and a positive Borel measure μ defined on the time–frequency space (see Definition 4.1). Although it is known that, a standard dual can always be associated with a given frame (even in the case of frames associated with measures; see [8] for more details), in the case of irregular Gabor systems, it is possible that no dual of Gabor type exists, contrary to the situation for "regular Gabor systems." We obtain a characterization for the possible duals of Gabor type, h, associated with a pair (g,μ), where g is in the Schwartz class in terms of the symplectic Fourier transform of the measure μ and of the short-time Fourier transform $\mathcal{V}_g h$ (Theorem 4.6). The characterization for Gabor duality obtained is also valid for a weaker definition of Gabor duality in which the dual h is allowed to be a tempered distribution and the expansion associated with it takes place in $\mathcal{S}'(\mathbb{R}^n)$ instead of $L^2(\mathbb{R}^n)$ (Theorem 4.4). In both situations, the characterization obtained implies, in particular, that the Fourier transform of the measure μ must be equal to a positive multiple of the Dirac mass at the origin on a neighborhood of the origin (Proposition 4.8), a property which is always destroyed by applying a non-zero local perturbation to the measure μ. Since the frame property itself is not so sensitive to local perturbations, it follows that any such small non-zero perturbation of a Gabor system admitting a dual of Gabor type will result in a Gabor frame admitting no such dual (even if the window is replaced by another window in the Schwartz class). As in Section 3, we again obtain a characterization for the uniqueness of a measure μ with the property that h is a dual of Gabor type for the pair (g,μ): the short-time Fourier transform $\mathcal{V}_g h$ cannot vanish simultaneously at any points (x,ν) and $(-x,-\nu)$ in the time–frequency space (Corollary 4.7). Using this characterization, we extend the results of Section 3 to dual systems and show that if g and h are both generalized Gaussian (associated with possibly different parameters) and $(g,h)_2 = 1$, then the only measure μ such that h is a dual of Gabor type for the pair (g,μ) is the trivial one, $\mu = 1$ (Theorem 4.12). A similar result holds for the extreme-value windows (Proposition 4.15). We also show that if g and h are both generalized Gaussian multiplied by polynomials, the possible measures μ allowing for Gabor duality between these windows have to be non-discrete, preventing thus discrete expansions in terms of such dual Gabor windows to exist (Theorem 4.14). Theorem 4.6 is used again, in the case where the window g is the one-dimensional Gaussian $g(t) = 2^{1/4}e^{-\pi t^2}$, to show that, if a dual of Gabor type exists for the pair (g,μ), then the symplectic Fourier transform of μ must be supported on a discrete set and that the discrete sets obtained in this way can be characterized as certain subsets of the zero sets of the entire functions $F(z)$ in the Bargmann–Fock space $\mathcal{F}^2(\mathbb{C})$ satisfying $F(0) = 1$ (Proposition 4.9).

In Section 5, we point out that, when applied to unweighted regular Gabor systems, where the sampling set Λ in the time–frequency space is a full-rank lattice, our main results translate into some well-known results of Gabor analysis such as the Ron–Shen or the Wexler–Raz identity (Theorems 5.1 and 5.2) and their generalizations (of course, with the restriction that at least one of the windows involved belong to the Schwartz class).

In the last section of this paper, we define the Beurling density of a positive Borel measure μ on $\mathbb{R}^{2n}$ and show that the existence of a dual of Gabor type for a Gabor system (g,μ), with g in the Schwartz class, implies that the Beurling density of μ exists and is equal to one (Theorem 6.3). As an immediate consequence, it follows that, if a weighted, discrete irregular Gabor system associated with such a window g form a Parseval frame for $L^2(\mathbb{R}^n)$, then the associate sampling set Λ in the time–frequency space must have a lower Beurling density at least equal to one (Corollary 6.4). Finally, we conclude this paper by displaying examples of discrete sampling sets Λ having arbitrary large Beurling density and having the property that the associated irregular Gabor system $\mathcal{G} := \{e^{2\pi i\nu t}g(t-x)\}_{(x,\nu)\in\Lambda}$, with g being any window in $\mathcal{S}(\mathbb{R}^n)$, does not admit a dual of Gabor type (Proposition 6.5).

We point out that the bracket $\langle\cdot,\cdot\rangle$, linear in both variables, will be used to denote the duality between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. We will use the notation $(\cdot,\cdot)_2$ for the usual inner product in $L^2(\mathbb{R}^n)$, which is thus linear in the first variable and anti-linear in the second. $\mathcal{C}_0^\infty(U)$ will denote the space of infinitely differentiable complex-valued functions with compact support in the open set U. If α, β are multi-indices in $\mathbb{N}^n$, we will write $\alpha \leqslant \beta$ (respectively $\alpha < \beta$) when $\alpha_i \leqslant \beta_i$, for each $i = 1, \ldots, n$ (respectively $\alpha_i \leqslant \beta_i$, for each $i = 1, \ldots, n$ with strict inequality for at least one i). Also, $|\alpha| = \sum_i \alpha_i$, $\binom{\beta}{\alpha} = \prod_i \binom{\beta_i}{\alpha_i}$ if $\alpha \leqslant \beta$, and, if $\xi \in \mathbb{R}^n$, $\xi^\alpha = \prod_i \xi_i^{\alpha_i}$.

2. Translation-bounded measures and the Bessel condition

The notion of translation bounded measure is a useful tool in the theory of quasicrystals (see e.g. [17]). It will also play an important role in our following discussion as it characterizes the measures for which the Bessel condition holds for Gabor systems with a window in the Schwartz class as we will prove in this section.

Definition 2.1. We say that a positive Borel measure μ on $\mathbb{R}^{2n}$ is *translation bounded* if for every compact $K \subset \mathbb{R}^{2n}$, there exists a constant $C > 0$ such that

$$\mu(K + x) \leqslant C, \quad \forall x \in \mathbb{R}^{2n}. \tag{2.1}$$

Clearly, a measure will be translation bounded if the condition above holds for a fixed compact set with non-empty interior.

Proposition 2.2. *Let $g \in L^2(\mathbb{R}^n)$ be a window function with $\|g\|_2 = 1$ and suppose that μ is a positive Borel measure on $\mathbb{R}^{2n}$ which satisfies the Bessel condition* (1.6)*. Then, μ is translation bounded. In particular, μ is a tempered measure on $\mathbb{R}^{2n}$, i.e. there exists an integer $M \geqslant 1$ such that*

$$\int_{\mathbb{R}^{2n}} \frac{1}{(1 + |x|^2 + |\nu|^2)^M} \, d\mu(x, \nu) < \infty.$$

Proof. Note first that $\mathcal{V}_g g(0, 0) = 1$ and thus, using the continuity of $\mathcal{V}_g g$, there exists $r > 0$ such that $|\mathcal{V}_g g(x, \nu)|^2 \geqslant \frac{1}{2}$ if $|x|^2 + |\nu|^2 \leqslant r^2$. Given any point $(x_0, \nu_0) \in \mathbb{R}^{2n}$, choose $f \in L^2(\mathbb{R}^n)$ of the form

$$f(t) = e^{2\pi i t \cdot \nu_0} g(t - x_0).$$

With this choice of f, we have

$$\begin{aligned} \mathcal{V}_g f(x, \nu) &= \int_{\mathbb{R}^n} e^{-2\pi i t \cdot (\nu - \nu_0)} g(t - x_0) \overline{g(t - x)} \, dt \\ &= e^{-2\pi i x_0 \cdot (\nu - \nu_0)} \int_{\mathbb{R}^n} e^{-2\pi i s \cdot (\nu - \nu_0)} g(s) \overline{g\big(s - (x - x_0)\big)} \, ds \\ &= e^{-2\pi i x_0 \cdot (\nu - \nu_0)} \mathcal{V}_g g(x - x_0, \nu - \nu_0). \end{aligned}$$

If $|x - x_0|^2 + |\nu - \nu_0|^2 \leqslant r^2$, we have thus

$$\big|\mathcal{V}_g f(x, \nu)\big|^2 = \big|\mathcal{V}_g g(x_0 - x, \nu - \nu_0)\big|^2 \geqslant \frac{1}{2}$$

and, letting $B_r(x_0,\nu_0)=\{(x,\nu)\in\mathbb{R}^{2n},\, |x-x_0|^2+|\nu-\nu_0|^2\leqslant r^2\}$, we deduce that

$$\frac{1}{2}\int\limits_{B_r(x_0,\nu_0)} 1\,d\mu(x,\nu)\leqslant\int\limits_{\mathbb{R}^{2n}}\bigl|\mathcal{V}_g f(x,\nu)\bigr|^2\,d\mu(x,\nu)\leqslant C_1.$$

This shows that

$$\mu\bigl(B_r(x_0,\nu_0)\bigr)\leqslant 2C_1,\quad \forall (x_0,\nu_0)\in\mathbb{R}^{2n},$$

from which the conclusion of the lemma easily follows. □

The condition (2.1) is, in general, not sufficient to guarantee that the Bessel condition (1.6) holds since it does not involve the window g. (See the paper [18] where sufficient conditions for the Bessel condition to hold are given in the case of unweighted discrete irregular Gabor frame.) For example, in the one-dimensional case, the classical Gabor system $\{e^{2\pi i l t} g(t-k)\}_{(k,l)\in\mathbb{Z}^2}$ corresponds to the measure $\mu=\sum_{(k,l)\in\mathbb{Z}^2}\delta_{(k,l)}$ which is certainly translation bounded, but the system does not generate a Bessel collection for certain windows $g\in L^2(\mathbb{R})$, since it is well known that a necessary and sufficient condition for this to happen is that the Zak transform of g be bounded a.e., while the Zak transform of an arbitrary function in $g\in L^2(\mathbb{R})$ could be, when restricted to the set I^2, any function in $L^2(I^2)$, where $I=[0,1]$. On the other hand, the following proposition shows that the condition (2.1) guarantees that the Bessel condition (1.6) holds for windows in the Schwartz class. Before stating it, we will need the following lemma [9, Lemma 11.3.3].

Lemma 2.3. *Let g_0, g, $\gamma\in\mathcal{S}(\mathbb{R}^n)$ be such that $(g,\gamma)_2\neq 0$ and let $f\in\mathcal{S}'(\mathbb{R}^n)$. Then, we have*

$$\bigl|\mathcal{V}_{g_0} f(x,\nu)\bigr|\leqslant\frac{1}{|(g,\gamma)_2|}\bigl(|\mathcal{V}_g f|*|\mathcal{V}_{g_0}\gamma|\bigr)(x,\nu),\quad (x,\nu)\in\mathbb{R}^{2n},\tag{2.2}$$

where $*$ denotes the convolution product on $\mathbb{R}^{2n}$.

Proposition 2.4. *Let μ be a positive, translation bounded Borel measure on $\mathbb{R}^{2n}$ and let $g\in\mathcal{S}(\mathbb{R}^n)$. Then, there exists a constant $C_1\geqslant 0$ such that* (1.6) *holds.*

Proof. We can assume that $g\neq 0$. Applying the inequality (2.2) in Lemma 2.3 with the functions $g_0=\gamma=g\in\mathcal{S}(\mathbb{R}^n)$ and $f\in L^2(\mathbb{R}^n)$, we obtain that

$$\bigl|\mathcal{V}_g f(x,\nu)\bigr|\leqslant C\bigl(|\mathcal{V}_g f|*|\mathcal{V}_g g|\bigr)(x,\nu),\quad (x,\nu)\in\mathbb{R}^{2n},\tag{2.3}$$

with $C=1/\|g\|_2^2$. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\bigl(|\mathcal{V}_g f|*|\mathcal{V}_g g|(x,\nu)\bigr)^2\\&=\Bigl(\int\limits_{\mathbb{R}^{2n}}\bigl|\mathcal{V}_g f(y,\omega)\bigr|\bigl|\mathcal{V}_g g(x-y,\nu-\omega)\bigr|\,dy\,d\omega\Bigr)^2\end{aligned}$$

$$\leqslant \int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(y,\omega)\right|^2 \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| dy\, d\omega \int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| dy\, d\omega$$

$$= C' \int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(y,\omega)\right|^2 \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| dy\, d\omega,$$

where $C' = \|\mathcal{V}_g g\|_{L^1} < \infty$ since $\mathcal{V}_g g \in \mathcal{S}(\mathbb{R}^{2n})$ by a result in [12] (see also [9, Theorem 11.2.5]). We can thus find, using (2.3), a constant C'' depending only on g such that

$$\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(x,\nu)\right|^2 d\mu(x,\nu)$$

$$\leqslant C'' \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(y,\omega)\right|^2 \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| dy\, d\omega \right) d\mu(x,\nu)$$

$$= C'' \int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(y,\omega)\right|^2 \left(\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| d\mu(x,\nu) \right) dy\, d\omega.$$

The translation boundedness of μ, i.e. (2.1), easily shows the existence of a constant $D > 0$ such that

$$\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g g(x-y,\nu-\omega)\right| d\mu(x,\nu) \leqslant D, \quad (y,\omega) \in \mathbb{R}^{2n}.$$

We deduce thus the existence of a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(x,\nu)\right|^2 d\mu(x,\nu) \leqslant C_1 \int_{\mathbb{R}^{2n}} \left|\mathcal{V}_g f(y,\omega)\right|^2 dy\, d\omega, \quad f \in L^2(\mathbb{R}^n),$$

and the conclusion follows from Theorem 1.1. □

3. A characterization for tight Gabor systems

Our next goal is to give a characterization for the positive Borel measures on $\mathbb{R}^{2n}$ satisfying the identity (1.4) and to derive some immediate consequences of this theorem. Note that many of the results in this section are given without proofs as they are particular cases of analogous results valid for dual systems, which are proved in Section 4. We first need the following definition.

Definition 3.1. If $h \in L^1(\mathbb{R}^{2n})$, we define its Fourier transform, $\mathcal{F}h$, by the formula

$$(\mathcal{F}h)(\xi,\eta) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(\xi\cdot x+\eta\cdot y)} h(x,y)\, dx\, dy, \quad (\xi,\eta) \in \mathbb{R}^{2n},$$

while the symplectic Fourier transform of h, $\mathcal{F}^S h$, is given by the formula

$$\left(\mathcal{F}^S h\right)(\xi,\eta) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(\eta\cdot x-\xi\cdot y)} h(x,y)\,dx\,dy, \quad (\xi,\eta)\in\mathbb{R}^{2n}.$$

Note that $(\mathcal{F}^S h)(\xi,\eta) = (\mathcal{F}h)(\eta,-\xi)$.

The definition of the Fourier transform and that of the symplectic Fourier transform are extended in the usual way to the elements of $\mathcal{S}'(\mathbb{R}^{2n})$: if $\sigma\in\mathcal{S}'(\mathbb{R}^{2n})$ and $\varphi\in\mathcal{S}(\mathbb{R}^{2n})$, we have

$$\langle\mathcal{F}\sigma,\varphi\rangle = \langle\sigma,\mathcal{F}\varphi\rangle, \qquad \left\langle\mathcal{F}^S\sigma,\varphi\right\rangle = \left\langle\sigma(\xi,\eta),\mathcal{F}\varphi(-\eta,\xi)\right\rangle.$$

The following characterization for our generalized tight Gabor frames associated with a measure is actually a particular case of a more general result valid for dual systems which will be proved in Section 4 (Theorem 4.6).

Theorem 3.2. *Let μ be a positive, translation bounded Borel measure on $\mathbb{R}^{2n}$ and let $g\in\mathcal{S}(\mathbb{R}^n)$. Then, the identity* (1.4) *holds for all $f\in L^2(\mathbb{R}^n)$ if and only if*

$$\left(\mathcal{F}^S\mu\right)(\mathcal{V}_g g) = \delta_{(0,0)} \quad \text{on } \mathbb{R}^{2n}. \tag{3.1}$$

Remark. The product on the left-hand side of (3.1) has to be understood as the product of a distribution in $\mathcal{S}'(\mathbb{R}^n)$ with a function in $\mathcal{S}(\mathbb{R}^n)$ and is thus a well-defined tempered distribution. Note that this product might no longer make sense if we assumed that the window was just in $L^2(\mathbb{R}^n)$ since the short-time Fourier transform $\mathcal{V}_g g$ might not necessarily be infinitely differentiable, but just continuous, and the symplectic Fourier transform of a positive, translation bounded Borel measure need not be a measure. For example, there exist real-valued functions in $L^\infty(\mathbb{R}^{2n})$ whose symplectic Fourier transforms are not measures (locally), and adding a small multiple of one of these to the function 1 will yield a density $f(t)$ for a measure $\mu = f(t)\,dt$ which is positive, translation bounded, and such that $\mathcal{F}^S(\mu)$ is not a measure.

Remark. We note that, for windows in $\mathcal{S}(\mathbb{R}^n)$, Theorem 1.1 follows immediately from the preceding theorem since $\mathcal{F}^S(1) = \delta_{(0,0)}$.

The following corollary provides a very simple criterion to determine whether or not the Lebesgue measure on $\mathbb{R}^{2n}$ (which we identify with the function 1) is the only measure satisfying (1.4) for all $f\in L^2(\mathbb{R}^n)$.

Corollary 3.3. *Let $g\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\|g\|_2 = 1$. Then, the measure $\mu = 1$ is the only translation-bounded positive Borel measure on $\mathbb{R}^{2n}$ satisfying the identity* (1.4) *for all $f\in L^2(\mathbb{R}^n)$ if and only if*

$$\mathcal{V}_g g(x,\nu)\neq 0, \quad \text{for all } (x,\nu)\in\mathbb{R}^{2n}. \tag{3.2}$$

Proof. Let $g\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\|g\|_2 = 1$. If (3.2) holds and μ is a translation-bounded measure satisfying (1.4), we have using (3.1) that

$$\left(\mathcal{F}^S(\mu-1)\right)\mathcal{V}_g g = 0 \quad \text{on } \mathbb{R}^{2n},$$

and thus $\mu = 1$. On the other hand, if, at some point $(x_0, \nu_0) \in \mathbb{R}^{2n}$, we have $\mathcal{V}_g g(x_0, \nu_0) = 0$, it follows that $\mathcal{V}_g g(-x_0, -\nu_0) = 0$ also, since

$$\mathcal{V}_g g(-x, -\nu) = e^{-2\pi i x\cdot\nu} \overline{\mathcal{V}_g g(x, \nu)}, \quad (x, \nu) \in \mathbb{R}^{2n}.$$

Therefore, using the identity

$$\mathcal{F}^S\big[\cos\big(2\pi(x\nu_0 - \nu x_0)\big)\big] = \frac{1}{2}(\delta_{(x_0,\nu_0)} + \delta_{(-x_0,-\nu_0)}),$$

we find that $\mu = 1 + \cos(2\pi(x\nu_0 - \nu x_0))$ is a translation-bounded measure different from 1 which satisfies (3.1) and the result follows from Theorem 3.2. □

Perhaps surprisingly, the *even* window functions g in the Schwartz class satisfying the uniqueness condition in the previous theorem which is equivalent to the non-vanishing of the function $\mathcal{V}_g g$ can be exactly characterized using a result of Hudson ([14]; see also [9, Theorem 4.4.1]): they must be "generalized Gaussians." Before stating this result, we need the following definitions.

Definition 3.4. A function $g \in L^2(\mathbb{R}^n)$ of the form

$$g(x) = e^{-Ax\cdot x + 2\pi b\cdot x + c}, \quad x \in \mathbb{R}^n, \tag{3.3}$$

where A is an $n \times n$ invertible matrix with complex entries and positive-definite real part $(A + A^*)/2$, where $b \in \mathbb{C}^n$ and $c \in \mathbb{C}$, is called a *generalized Gaussian*. We can assume that A is symmetric in this definition since the function g is unchanged when A is replaced by $(A + A^t)/2$, where A^t denotes the transpose of A.

The Wigner distribution, which was first introduced by Wigner, plays an important role in quantum mechanics.

Definition 3.5. The *Wigner distribution* of a function $f \in L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{W}f(x, \nu) = \int_{\mathbb{R}^n} f(x + t/2)\overline{f(x - t/2)} e^{-2\pi i \nu\cdot t}\, dt, \quad (x, \nu) \in \mathbb{R}^{2n}. \tag{3.4}$$

Note that $\mathcal{W}f$ is real-valued.

Definition 3.6. The *ambiguity function* of a function $f \in L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{A}f(x, \nu) = \int_{\mathbb{R}^n} f(t + x/2)\overline{f(t - x/2)} e^{-2\pi i \nu\cdot t}\, dt, \quad (x, \nu) \in \mathbb{R}^{2n}. \tag{3.5}$$

It satisfies $\mathcal{A}f(-x, -\nu) = \overline{\mathcal{A}f(x, \nu)}$ and is related to the short-time Fourier transform via the formula

$$\mathcal{A}g(x, \nu) = e^{i\pi x\cdot\nu} \mathcal{V}_g g(x, \nu), \quad (x, \nu) \in \mathbb{R}^{2n}. \tag{3.6}$$

We should point out that, although the formulas defining the Wigner distribution and the ambiguity function are very similar, these two functions can be very different. In particular, the sets where they vanish might be completely unrelated. (See the comments after Theorem 3.10.)

The following result of Hudson completely characterizes as generalized Gaussians the functions in $L^2(\mathbb{R}^n)$ with a non-vanishing Wigner distribution.

Theorem 3.7. *(See [14].) Let $f \in L^2(\mathbb{R}^n)$. Then, $\mathcal{W}f(x,\nu) > 0$ for all $(x,\nu) \in \mathbb{R}^{2n}$ if and only if f is a generalized Gaussian of the form* (3.3).

As an immediate consequence of this last result and of Corollary 3.3, we have the following.

Proposition 3.8. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ be even* (*i.e.* $g(-x) = g(x)$, *for all $x \in \mathbb{R}^n$*) *and satisfy $\|g\|_2 = 1$. Then, the following are equivalent*:

(a) *The measure $\mu = 1$ is the only translation-bounded positive Borel measure on $\mathbb{R}^{2n}$ satisfying the identity* (1.4) *for all $f \in L^2(\mathbb{R}^n)$.*
(b) *$\mathcal{V}_g g(x,\nu) \neq 0$, for all $(x,\nu) \in \mathbb{R}^{2n}$.*
(c) *$g(x) = Ce^{-Ax\cdot x}$, where $C \neq 0$ is a constant and $A \in GL(n,\mathbb{C})$ is an $n \times n$ invertible matrix with positive-definite real part.*

Proof. The equivalence of (a) and (b) is the statement of Corollary 3.3. Furthermore, note that for any $g \in L^2(\mathbb{R}^n)$ and g is even, we have

$$\mathcal{W}g(x/2,\nu/2)2^{-n} = \mathcal{A}g(x,\nu), \quad (x,\nu) \in \mathbb{R}^{2n}, \tag{3.7}$$

and, in particular, $\mathcal{W}g(0,0) = \mathcal{A}g(0,0)2^n = 2^n > 0$. Since $\mathcal{W}g$ is real, the fact that $\mathcal{W}g$ does not vanish means that $\mathcal{W}g > 0$ on $\mathbb{R}^{2n}$. By Hudson's result [14] and the identity (3.6), the fact that $\mathcal{V}_g g$ is never zero is then equivalent to g being a generalized Gaussian of the form (c), using the evenness of g. $\square$

In the case of an odd window, we always have the non-uniqueness of the measure μ.

Proposition 3.9. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ be odd* (*i.e.* $g(-x) = -g(x)$, *for all $x \in \mathbb{R}^n$*) *and satisfy $\|g\|_2 = 1$. Then, there is more then one translation-bounded Borel measure μ on $\mathbb{R}^{2n}$ satisfying the identity* (1.4) *for all $f \in L^2(\mathbb{R}^n)$.*

Proof. If g is odd, we have

$$\mathcal{W}g(x/2,\nu/2)2^{-n} = -\mathcal{A}g(x,\nu) = -e^{i\pi x\cdot\nu}\mathcal{V}_g g(x,\nu), \quad (x,\nu) \in \mathbb{R}^{2n}, \tag{3.8}$$

and, in particular, $\mathcal{W}g(0,0) = -\mathcal{A}g(0,0)2^n = -2^n < 0$. On the other hand, it is known (see, e.g. [9, Lemma 4.3.6]) that, for any $f \in L^2(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^{2n}} \mathcal{W}f(x,\nu)\,dx\,d\nu = \|f\|_2^2.$$

Since $\mathcal{W}g$ is real-valued, $\mathcal{W}g$ must thus take on some positive values. The continuity of $\mathcal{W}g$ together with the intermediate value theorem shows then that $\mathcal{W}g(x_0, \nu_0) = 0$ for some $(x_0, \nu_0) \in \mathbb{R}^{2n}$ and the conclusion follows from Corollary 3.3. □

Our next result shows that, whether they are even or not, the functions in $L^2(\mathbb{R}^n)$ with a positive Wigner distribution, which by Hudson's theorem are exactly the generalized Gaussians of the form (3.3), have a non-vanishing ambiguity function.

Theorem 3.10. *Let g be a generalized Gaussian of the form* (3.3)*. Then, $\mathcal{V}_g g(x, \nu) \neq 0$ for all $(x, \nu) \in \mathbb{R}^{2n}$ and if, in addition $\|g\|_2 = 1$, the measure $\mu = 1$ is the only translation-bounded Borel measure for which* (1.4) *holds. In particular, no weighted irregular tight frame for $L^2(\mathbb{R}^n)$ can be constructed using a generalized Gaussian as a window function.*

Proof. The fact that $\mathcal{V}_g g$ does not vanish follows from Lemma 4.11 (with $h = g$) which will be proved later. The remainder of the statement follows from Corollary 3.3 and the identity (1.3). □

For a general window $g \in \mathcal{S}(\mathbb{R}^n)$, the positivity of the associated Wigner distribution implies thus the non-vanishing of the ambiguity function or, equivalently, that of $\mathcal{V}_g g$. For windows that are either even or odd, the relations (3.7) and (3.8) between the ambiguity function and the Wigner distribution show that the non-vanishing of one is equivalent to that of the other. This suggests that, perhaps, these two properties are equivalent in general and that the generalized Gaussians are the only windows for which the ambiguity function does not vanish. However, for windows without (even or odd) symmetry this is not necessarily the case. For example, in one dimension, if $g(x) = e^{-\pi x}\chi_{[0,\infty)}(x)$, we have

$$\mathcal{A}(g)(x, \nu) = \frac{e^{-\pi|x|}e^{-i\pi|x|\nu}}{2\pi(1 + i\nu)} \neq 0, \quad (x, \nu) \in \mathbb{R}^2,$$

while

$$\mathcal{W}(g)(x, \nu) = e^{-2\pi x}\frac{\sin(4\pi x\nu)}{\pi\nu}\chi_{[0,\infty)}(x), \quad (x, \nu) \in \mathbb{R}^2,$$

vanishes for all $x \leqslant 0$. Of course, the previous window belongs to $L^2(\mathbb{R})$ but not to the Schwartz class. Nevertheless, as we will show next, counterexamples in the Schwartz class can also be constructed. The so-called "extreme value" window function appears as a density function in probability theory and is defined by

$$\psi(t) = e^{t - e^t}, \quad t \in \mathbb{R}.$$

More generally, we consider the functions $\psi_{k,m}$ defined by

$$\psi_{k,m} = e^{kt - me^t}, \quad t \in \mathbb{R},$$

where $k, m > 0$ are parameters. Clearly, $\psi_{k,m}$ decays exponentially at $-\infty$ and super-exponentially at ∞. Furthermore, since

$$\psi'_{k,m}(t) = \left(k - me^t\right)e^{kt-me^t} = ke^{kt-me^t} - me^{(k+1)t-me^t},$$

it follows, by induction, that the rth derivative of $\psi_{k,m}$ is a linear combination of $\psi_{k+j,m}$, $j = 0, \ldots, r$, and thus has also exponential decay at $\pm\infty$. Hence, $\psi_{k,m} \in \mathcal{S}(\mathbb{R})$. It is known that the Fourier transform of $\psi_{k,m}$ can be expressed in terms of the Gamma function:

$$\mathcal{F}\psi_{k,m}(\xi) = m^{-k+2\pi i\xi}\Gamma(k - 2\pi i\xi), \quad \xi \in \mathbb{R}.$$

The corresponding ambiguity function can also be expressed in terms of the Gamma function which yields the analogue of Theorem 3.10 for the normalized windows $\psi_{k,m}$. This result is a particular case of a more general result for dual systems (Proposition 4.15) which will be proved later.

Proposition 3.11. *We have*

$$\mathcal{A}(\psi_{k,m})(x, \nu) = \left[m\left(e^{x/2} + e^{-x/2}\right)\right]^{-2k+2\pi i\nu}\Gamma(2k - 2\pi i\nu).$$

In particular,

(a) $\mathcal{A}(\psi_{k,m})$ *does not vanish anywhere.*
(b) *If* $g(t) = (2m)^k(\Gamma(2k))^{-1/2}\psi_{k,m}$, *we have* $\|g\|_2 = 1$ *and the measure* $\mu = 1$ *is the only translation-bounded Borel measure for which* (1.4) *holds.*

Note that the Wigner distribution of $\psi_{k,m}$ has to vanish somewhere in the time–frequency plane by Hudson's theorem.

Even in the case where the measure μ in (1.4) is not unique, it is possible that no discrete measure is solution of the problem which then again prevents the existence of discrete, weighted tight irregular Gabor frames associated with the given window. This will be the case if, for example, the zero set of $\mathcal{V}_g g$ is compact as we prove next.

Proposition 3.12. *Let* $g \in \mathcal{S}(\mathbb{R}^n)$ *satisfy* $\|g\|_2 = 1$ *and suppose that the set*

$$\left\{(x, \nu) \in \mathbb{R}^{2n}, \mathcal{V}_g g(x, \nu) = 0\right\}$$

is a compact subset of $\mathbb{R}^{2n}$. *Then, any positive Borel measure* μ *satisfying the identity* (1.4) *must be absolutely continuous with respect to the Lebesgue measure, i.e.* $d\mu = f(t)\,dt$. *Furthermore,* $f(t)$ *is the restriction to* $\mathbb{R}^n$ *of an entire function of exponential type on* $\mathbb{C}^n$.

Proof. Using (3.1) and our assumption, we deduce that $\mathcal{F}^S\mu$, and thus also $\mathcal{F}\mu$, has compact support. The result follows then immediately from the Paley–Wiener–Schwartz theorem, see [24]. $\square$

Example. If we let $g(t) = te^{-\pi t^2}$, for $t \in \mathbb{R}$, we have then

$$\mathcal{A}g(x,\nu) = -\frac{\sqrt{2}}{8}e^{-\frac{\pi}{2}(x^2+\nu^2)}\left(x^2+\nu^2-\frac{1}{\pi}\right), \quad (x,\nu)\in\mathbb{R}^2,$$

and the previous result applies, since if μ is any positive measure satisfying the identity (1.4), the support of $\mathcal{F}^S\mu$ will then be contained in the circle centered at the origin and of radius $\frac{1}{\sqrt{\pi}}$ in the time–frequency plane.

The compactness of the zero set of $\mathcal{V}_g g$ implies the absolute continuity of the measure μ in the previous proposition but, in fact, much less is required to prevent that measure from being discrete as the following particular case of Theorem 4.14 shows.

Theorem 3.13. *Let $g(x) = p(x)e^{-\pi Ax\cdot x+\pi a\cdot x}$ where A is a symmetric matrix in $GL(n,\mathbb{C})$ with positive definite real part, $a \in \mathbb{C}^n$ and $p(x)$ is a polynomial in n variables chosen so that $\|g\|_2 = 1$. Then, no discrete, irregular weighted tight Gabor frame can be constructed using $g(x)$ as a window function.*

Note that if the "tightness" restriction is removed, regular Gabor frames can be constructed with windows such as those appearing in the previous theorem. For example, Gröchenig and Lyubarskiĭ [11] have recently showed that the regular Gabor system associated with the Hermite function $H_n(t) = e^{\pi t^2}(\frac{d}{dt})^n(e^{-\pi t^2})$ and a lattice $A(\mathbb{Z}^2)$, where A is an invertible 2×2 real matrix, forms a frame for $L^2(\mathbb{R})$ as long as $|\det(A)| < (n+1)^{-1}$.

Our next result shows that the property of a translation-bounded measure μ to satisfy the identity (1.4) for a certain window function is destroyed by any local perturbation of that measure.

Proposition 3.14. *Let μ be a translation-bounded measure on $\mathbb{R}^{2n}$ which satisfies the identity* (1.4) *for a certain window $g_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\|g_0\|_2 = 1$. Then, there exists $r > 0$, such that*

$$\mathcal{F}\mu = \delta_{(0,0)} \quad \text{on } B_r, \tag{3.9}$$

where B_r denotes the open ball of radius r centered at the origin in $\mathbb{R}^{2n}$. In particular, if K is a compact subset of $\mathbb{R}^{2n}$ and ρ is a translation-bounded measure on $\mathbb{R}^{2n}$ different from μ such that $\mu = \rho$ on $\mathbb{R}^{2n}\setminus K$, then ρ fails to satisfy the identity (1.4) *for any window $g \in \mathcal{S}(\mathbb{R}^n)$ with $\|g\|_2 = 1$.*

Proof. If μ satisfies (1.4) with the window g_0, then (3.9) follows immediately from the identity (3.1) in Theorem 3.2 since $\mathcal{V}_{g_0}g_0(0,0) = 1$. If ρ is as above and satisfies (1.4), then the measure $\mu - \rho$ is translation-bounded, compactly supported and has a Fourier transform that vanishes on B_s, for some $s > 0$. By the Paley–Wiener–Schwartz theorem [24], $\mathcal{F}(\mu-\rho)$ is the restriction to $\mathbb{R}^{2n}$ of an entire function of exponential type on $\mathbb{C}^{2n}$ and has thus to vanish identically if it vanishes on B_s. This contradicts the fact that $\rho \neq \mu$ and proves our claim. □

4. Dual windows of Gabor type

In this section we consider the problem of constructing dual Gabor windows associated with a Gabor window belonging to the Schwartz class. The possibility of constructing a dual Gabor window different than the original one is quite important since, as was seen in the previous section, the generalized Gabor expansions that can associated with certain windows (such as the generalized Gaussians) are quite limited if one insists on self-duality. In the following, we do not require the dual window to belong to the Schwartz class but we allow it to be a priori any function in $L^2(\mathbb{R}^n)$ as in our next definition or even any tempered distribution as in Theorem 4.4.

Definition 4.1. Given $g \in \mathcal{S}(\mathbb{R}^n)$ and a positive, translation-bounded measure μ on $\mathbb{R}^{2n}$, we say that the function $h \in L^2(\mathbb{R}^n)$ is a dual window of Gabor type for the pair (g, μ) if h satisfies the Bessel condition (1.6) (with g replaced with h) with respect to μ and if we have

$$\int\limits_{\mathbb{R}^{2n}} \mathcal{V}_g f_1(x,\nu)\overline{\mathcal{V}_h f_2(x,\nu)}\, d\mu(x,\nu) = \int\limits_{\mathbb{R}^n} f_1(t)\overline{f_2(t)}\, dt, \qquad f_1, f_2 \in L^2(\mathbb{R}^n). \tag{4.1}$$

Note that, under the conditions of the previous definition, the left-hand side of Eq. (4.1) is well defined since g satisfies the Bessel condition with respect to μ by Proposition 2.4. It is also worth mentioning that if a dual of Gabor type h exists for the pair (g, μ), there exists a constant $C > 0$ such that $\|f\|_2 \leqslant C\|\mathcal{V}_g f\|_{2,\mu}$ holds for all $f \in L^2(\mathbb{R}^n)$, by applying the Cauchy–Schwarz inequality (with $f_1 = f_2$) to the left-hand side of (4.1). The system (g, μ) is thus a "generalized frame" in the sense of the theory developed in [8].

Formula (4.1) leads directly to the following generalized Gabor expansion formulas in the case where h is a dual window for the pair (g, μ):

$$\begin{aligned} f(t) &= \int\limits_{\mathbb{R}^{2n}} \mathcal{V}_g f(x,\nu) e^{2\pi i\nu\cdot t} h(t-x)\, d\mu(x,\nu) \\ &= \int\limits_{\mathbb{R}^{2n}} \mathcal{V}_h f(x,\nu) e^{2\pi i\nu\cdot t} g(t-x)\, d\mu(x,\nu), \qquad f \in L^2(\mathbb{R}^n). \end{aligned}$$

Lemma 4.2. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $h \in \mathcal{S}'(\mathbb{R}^n)$. Define*

$$(\mathcal{K}\varphi)(x,\nu) = \big\langle g(t) \otimes h(s), e^{-2\pi i\nu\cdot(t-s)}\varphi(t+x, s+x)\big\rangle, \qquad \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big).$$

Then, $\mathcal{K}$ is a continuous mapping from $\mathcal{S}(\mathbb{R}^{2n})$ to itself.

Proof. Note that if $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$\big\langle g(t) \otimes h(s), \varphi(t,s)\big\rangle = \big\langle h(s), \big\langle g(t), \varphi(t,s)\big\rangle\big\rangle$$

where $k(s) = \langle g(t), \varphi(t,s)\rangle$ belongs to $\mathcal{S}(\mathbb{R}^n)$. For any integer $m \geqslant 0$, let

$$\|\psi\|_{\mathcal{S}_m} = \sup_{|\alpha|\leqslant m} \big\|D^\alpha\psi(x)\big(1+|x|^2\big)^m\big\|_\infty, \qquad \psi \in \mathcal{S}\big(\mathbb{R}^n\big).$$

Define

$$\mathcal{R}[\varphi](x,\nu,s) = \big\langle g(t), e^{-2\pi i\nu\cdot(t-s)}\varphi(t+x, s+x)\big\rangle, \qquad \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big).$$

If β_1, β_2 and γ are multi-indices in $\mathbb{N}^n$, we have

$$\begin{aligned}
&D_x^{\beta_1} D_\nu^{\beta_2} D_s^{\gamma} \mathcal{R}[\varphi](x,\nu,s)\\
&\quad = D_x^{\beta_1} D_\nu^{\beta_2} \sum_{\gamma_1 \leqslant \gamma} \binom{\gamma}{\gamma_1} (2\pi i \nu)^{\gamma-\gamma_1} \big\langle g(t), e^{-2\pi i \nu\cdot(t-s)} D^{(0,\gamma_1)} \varphi(t+x, s+x)\big\rangle\\
&\quad = D_x^{\beta_1} \Bigg\{ \sum_{\gamma_1 \leqslant \gamma} \binom{\gamma}{\gamma_1} (2\pi i)^{\gamma-\gamma_1}\\
&\qquad \times \sum_{\substack{\delta \leqslant \beta_2\\ \delta \leqslant \gamma-\gamma_1}} \binom{\beta_2}{\delta} (\nu)^{\gamma-\gamma_1-\delta} \big\langle g(t), e^{-2\pi i \nu\cdot(t-s)} \big(2\pi i (s-t)\big)^{\beta_2-\delta} D^{(0,\gamma_1)} \varphi(t+x, s+x)\big\rangle \Bigg\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&D_x^{\beta_1} \big\{ \big\langle g(t), \big(2\pi i (s-t)\big)^{\nu-\delta} e^{-2\pi i \nu\cdot(t-s)} D^{(0,\gamma_1)} \varphi(t+x, s+x)\big\rangle \big\}\\
&\quad = \big\langle g(t), \big(2\pi i (s-t)\big)^{\nu-\delta} e^{-2\pi i \nu\cdot(t-s)} D^{(\beta_1,\gamma_1)} \varphi(t+x, s+x)\big\rangle\\
&\qquad + \big\langle g(t), \big(2\pi i (s-t)\big)^{\nu-\delta} e^{-2\pi i \nu\cdot(t-s)} D^{(0,\beta_1+\gamma_1)} \varphi(t+x, s+x)\big\rangle\\
&\quad = \big\langle g(t), e^{-2\pi i \nu\cdot(t-s)} \psi(t+x, s+x)\big\rangle,
\end{aligned}$$

where

$$\psi(t,s) = \big(2\pi i (s-t)\big)^{\nu-\delta} \big\{ D^{(\beta_1,\gamma_1)} \varphi(t,s) + D^{(0,\beta_1+\gamma_1)} \varphi(t,s)\big\}$$

and the mapping $\varphi \mapsto \psi$ is continuous from $\mathcal{S}(\mathbb{R}^{2n})$ to itself. It follows that

$$D_x^{\beta_1} D_\nu^{\beta_2} D_s^{\gamma} \mathcal{R}[\varphi](x,\nu,s) = \sum_{\sigma \leqslant \gamma} c_\sigma (\nu)^\sigma \big\langle g(t), e^{-2\pi i \nu\cdot(t-s)} \psi_\sigma(t+x, s+x)\big\rangle \tag{4.2}$$

where, for each multi-index σ, c_σ is a complex constant and $\psi_\sigma \in \mathcal{S}(\mathbb{R}^{2n})$ with the linear mapping $\varphi \mapsto \psi_\sigma$ from $\mathcal{S}(\mathbb{R}^{2n})$ to itself being continuous. Note that

$$\begin{aligned}
&(\nu)^\sigma \big\langle g(t), e^{-2\pi i \nu\cdot(t-s)} \psi_\sigma(t+x, s+x)\big\rangle\\
&\quad = \frac{1}{(-2\pi i)^{|\sigma|}} \int_{\mathbb{R}^n} g(t) \psi_\sigma(t+x, s+x) D_t^\sigma \big\{ e^{-2\pi i \nu\cdot(t-s)} \big\}\, dt.
\end{aligned}$$

Since, for fixed x and s, the function $t \mapsto g(t)\psi_\sigma(t+x, s+x)$ belongs to $\mathcal{S}(\mathbb{R}^n)$, integration by parts shows that the last expression can be written as

$$\begin{aligned}
&\frac{1}{(2\pi i)^{|\sigma|}} \int_{\mathbb{R}^n} D_t^\sigma \big\{ g(t)\psi_\sigma(t+x, s+x)\big\} e^{-2\pi i \nu\cdot(t-s)} dt\\
&\quad = \frac{1}{(2\pi i)^{|\sigma|}} \sum_{\sigma_1 \leqslant \sigma} \binom{\sigma}{\sigma_1} \int_{\mathbb{R}^n} D_t^{\sigma_1} g(t) D_t^{\sigma-\sigma_1} \psi_\sigma(t+x, s+x) e^{-2\pi i \nu\cdot(t-s)}\, dt.
\end{aligned}$$

Since g and each ψ_σ belong to the Schwartz class, we can find, for any integer $m > n/2$, a constant C (depending on g) such that the modulus of the previous expression is bounded by

$$C \int_{\mathbb{R}^n} \frac{1}{(1+|t|^2)^m} \frac{1}{(1+|t+x|^2+|s+x|^2)^m}\, dt\, \|\varphi\|_{\mathcal{S}_k}$$

where $k = k(\beta_1, \beta_2, \gamma, m, g)$. Using Peetre's inequality (see [23, Lemma 1.18]), this last expression is itself bounded by

$$C_1 \int_{\mathbb{R}^n} \frac{1}{(1+|t|^2)^m} \frac{1}{(1+|t|^2+|s|^2)^m} \frac{1}{(1+|x|^2)^m}\, dt\, \|\varphi\|_{\mathcal{S}_k}$$
$$\leqslant C_2 \frac{1}{(1+|s|^2)^m} \frac{1}{(1+|x|^2)^m} \|\varphi\|_{\mathcal{S}_k}.$$

Using (4.2) and the previous estimate, it is then easy to see that, for each integer $m \geqslant 0$, there is a constant C_m such that, for $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\left(1+|x|^2+|\nu|^2\right)^m \left|D_x^{\beta_1} D_\nu^{\beta_2} D_s^{\gamma} \mathcal{R}[\varphi](x,\nu,s)\right| \leqslant C_m \left(1+|s|^2\right)^{-m} \|\varphi\|_{\mathcal{S}_k}, \tag{4.3}$$

if $\max\{|\beta_1|, |\beta_2|, |\gamma|\} \leqslant m$, where $k = k(m, g)$. Since $h \in \mathcal{S}'(\mathbb{R}^n)$, there exists $C_0 > 0$ and an integer $m_0 \geqslant 0$ such that

$$\left|\left\langle h(s), \psi(s)\right\rangle\right| \leqslant C_0 \|\psi\|_{\mathcal{S}_{m_0}}, \quad \psi \in \mathcal{S}\left(\mathbb{R}^n\right). \tag{4.4}$$

Therefore, for any multi-indices β_1, β_2 in $\mathbb{N}^n$, we have

$$D_x^{\beta_1} D_\nu^{\beta_2} (\mathcal{K}\varphi)(x,\nu) = D_x^{\beta_1} D_\nu^{\beta_2} \left\langle h(s), \mathcal{R}[\varphi](x,\nu,s)\right\rangle$$
$$= \left\langle h(s), D_x^{\beta_1} D_\nu^{\beta_2} \mathcal{R}[\varphi](x,\nu,s)\right\rangle$$

and, using the inequalities (4.3) and (4.4), we have thus, if $m \geqslant m_0$ and $\max\{|\beta_1|, |\beta_2|, |\gamma|\} \leqslant m$, that

$$\left|D_x^{\beta_1} D_\nu^{\beta_2} (\mathcal{K}\varphi)(x,\nu)\right| \leqslant C_0 \sup_{\substack{s \in \mathbb{R}^n \\ |\gamma| \leqslant m_0}} \left|D_x^{\beta_1} D_\nu^{\beta_2} D_s^{\gamma} \mathcal{R}[\varphi](x,\nu,s)\left(1+|s|^2\right)^{m_0}\right|$$
$$\leqslant C'_m \left(1+|x|^2+|\nu|^2\right)^{-m} \|\varphi\|_{\mathcal{S}_k},$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, where $k = k(\beta_1, \beta_2, m, g, h)$. This proves our claim. $\square$

The following lemma will be needed. It offers a slight improvement to [9, Theorem 11.2.3].

Lemma 4.3. *Let $g \in \mathcal{S}(\mathbb{R}^n)$. Then, if $h \in \mathcal{S}'(\mathbb{R}^n)$, the function $\mathcal{V}_g h(x,\nu)$ belongs to $\mathcal{O}_M(\mathbb{R}^{2n})$, i.e. for any multi-indices α, $\beta \in \mathbb{R}^{2n}$, there is a constant $C(\alpha,\beta) > 0$ and an integer $m(\alpha,\beta)$ such that*

$$\left|D_x^{\alpha} D_\nu^{\beta} \mathcal{V}_g h(x,\nu)\right| \leqslant C(\alpha,\beta)\left(1+|x|^2+|\nu|^2\right)^{m(\alpha,\beta)}, \quad (x,\nu) \in \mathbb{R}^{2n}. \tag{4.5}$$

Furthermore, if B is a bounded set in $\mathcal{S}'(\mathbb{R}^n)$, the constants $C(\alpha,\beta)$ and $m(\alpha,\beta)$ can be chosen such that the estimate (4.5) *holds uniformly for all $h \in B$.*

Proof. Note that if B is a bounded set in $\mathcal{S}'(\mathbb{R}^n)$, there exists a constant $C \geqslant 0$ and an integer $m \geqslant 0$ such that

$$\left|\langle h,\varphi\rangle\right| \leqslant C\|\varphi\|_{\mathcal{S}_m}, \quad \varphi \in \mathcal{S}\left(\mathbb{R}^n\right),\ h \in B.$$

In the case where $\alpha=\beta=0$ and B consists of a single element, the estimate (4.5) is exactly the statement of Theorem 11.2.3 in [9], but the proof is the same if B is a bounded set in $\mathcal{S}'(\mathbb{R}^n)$. In general, we have

$$\begin{aligned} D_x^\alpha D_\nu^\beta \mathcal{V}_g h(x,\nu) &= (-1)^{|\alpha|}\left\langle h(t), (-2\pi i t)^\beta e^{-2\pi i \nu\cdot t} D^\alpha \bar{g}(t-x)\right\rangle \\ &= (-1)^{|\alpha|}(-2\pi i)^{|\beta|}\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma} x^{\beta-\gamma}\left\langle h(t), (t-x)^\gamma e^{-2\pi i\nu\cdot t} D^\alpha \bar{g}(t-x)\right\rangle \\ &= (-1)^{|\alpha|}(-2\pi i)^{|\beta|}\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma} x^{\beta-\gamma}\mathcal{V}_{\psi_{\alpha,\gamma}} h(x,\nu), \end{aligned}$$

where $\psi_{\alpha,\gamma}(t) = t^\gamma D^\alpha \bar{g}(t)$, and the result follows immediately from the previous case. □

We prove next a version of our dual window characterization in which the functions to be expanded are in the Schwartz class instead of being square-integrable but where we allow the dual window to be a tempered distribution and the expansion takes place in the space of tempered distributions. This situation occurs, for example, in the case of the one-dimensional regular Gabor system associated with the lattice $\mathbb{Z}^2$, where the Balian–Low theorem prevents any function nicely localized in the time–frequency plane, and in particular, any Schwartz function, to form a frame for $L^2(\mathbb{R})$ and thus to admit a Gabor dual in $L^2(\mathbb{R})$ in the sense of Definition 4.1. However, the weaker duality defined by (4.6) can still occur. An example of such duality in the distributional sense in the case where the window is a Gaussian can be found in the paper [15] by Janssen.

Theorem 4.4. *Let μ be a tempered positive Borel measure on $\mathbb{R}^{2n}$. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and let $h \in \mathcal{S}'(\mathbb{R}^n)$. Then, the identity*

$$\int_{\mathbb{R}^{2n}} \mathcal{V}_g\psi_1(x,\nu)\overline{\mathcal{V}_h\psi_2(x,\nu)}\,d\mu(x,\nu) = \int_{\mathbb{R}^n}\psi_1(t)\overline{\psi_2(t)}\,dt, \quad \psi_1,\psi_2 \in \mathcal{S}\left(\mathbb{R}^n\right), \tag{4.6}$$

holds if and only if

$$\left(\mathcal{F}^S\mu\right)(\mathcal{V}_g h) = \delta_{(0,0)} \quad \text{on } \mathbb{R}^{2n}. \tag{4.7}$$

Proof. We first define a tempered distribution T on $\mathbb{R}^{2n}$ by the formula

$$\left\langle T(t,s),\varphi(t,s)\right\rangle = \int_{\mathbb{R}^{2n}} \left\{\left\langle\overline{g(t)}\otimes h(s), e^{-2\pi i\nu\cdot(t-s)}\varphi(t+x,s+x)\right\rangle\right\} d\mu(x,\nu),$$

for each $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$. Note that T is a well-defined tempered distribution by Lemma 4.2. It is also clear that, if $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^{2n}} \mathcal{V}_g\psi_1(x,\nu)\overline{\mathcal{V}_h\psi_2(x,\nu)}\,d\mu(x,\nu) = \big\langle T(t,s), \psi_1(t)\otimes\overline{\psi_2(s)}\big\rangle.$$

If (4.6) holds, we have thus

$$\big\langle T(t,s), \psi_1(t)\otimes\overline{\psi_2(s)}\big\rangle = \int_{\mathbb{R}^n} \psi_1(t)\overline{\psi_2(t)}\,dt, \quad \psi_1,\psi_2 \in \mathcal{S}\big(\mathbb{R}^n\big).$$

Defining the tempered distribution ρ on $\mathbb{R}^{2n}$ by the formula

$$\big\langle \rho(t,s), \varphi(t,s)\big\rangle = \int_{\mathbb{R}^n} \varphi(t,t)\,dt, \quad \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big),$$

and using the density in $\mathcal{S}(\mathbb{R}^{2n})$ of the span of the functions of the form $\psi_1(t)\otimes\overline{\psi_2(s)}$, where $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$, we deduce that $T = \rho$. The change of variable $u = s - t$, $w = s$ induces a transformation Φ from $\mathcal{S}'(\mathbb{R}^{2n})$ to itself defined by

$$\big\langle \Phi(\sigma)(u,w), \varphi(u,w)\big\rangle = \big\langle \sigma(t,s), \varphi(s-t,s)\big\rangle, \quad \sigma \in \mathcal{S}'\big(\mathbb{R}^{2n}\big),\ \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big).$$

With this change of variable, we have

$$\big\langle \Phi(T)(u,w), \varphi(u,w)\big\rangle = \int_{\mathbb{R}^{2n}} \big\{\big\langle \overline{g(t)}\otimes h(s), e^{-2\pi i\nu\cdot(t-s)}\varphi(s-t,s+x)\big\rangle\big\}\,d\mu(x,\nu),$$

for each $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, while

$$\big\langle \Phi(\rho)(u,w), \varphi(u,w)\big\rangle = \int_{\mathbb{R}^n} \varphi(0,w)\,dw,$$

for each such φ, showing that $\Phi(\rho)(u,w) = \delta_0(u)\otimes 1(w)$. Denoting by $\mathcal{F}_2$ the Fourier transform with respect to the second variable w and defined by the formula

$$\mathcal{F}_2\varphi(u,\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot w}\varphi(u,w)\,dw, \quad \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big),$$

for functions in the Schwartz class and, by duality,

$$\langle \mathcal{F}_2\sigma, \varphi\rangle = \langle \sigma, \mathcal{F}_2\varphi\rangle, \quad \varphi \in \mathcal{S}\big(\mathbb{R}^{2n}\big),\ \sigma \in \mathcal{S}'\big(\mathbb{R}^{2n}\big),$$

for tempered distributions on $\mathbb{R}^{2n}$, we have, that

$$\begin{aligned}
&\big\langle \overline{g(t)} \otimes h(s), e^{-2\pi i\nu\cdot(t-s)}\mathcal{F}_2\varphi(s-t, s+x)\big\rangle \\
&\quad = \big\langle h(s), \big\langle \overline{g(t)} e^{-2\pi i\nu\cdot(t-s)}\mathcal{F}_2\varphi(s-t, s+x)\big\rangle\big\rangle \\
&\quad = \Big\langle h(s), \iint_{\mathbb{R}^{2n}} \overline{g(t)} e^{-2\pi i\nu\cdot(t-s)} e^{-2\pi i(s+x)\cdot\xi}\varphi(s-t,\xi)\,d\xi\,dt\Big\rangle.
\end{aligned}$$

After the change of variables $s-t=u$, this last expression can be written as

$$\Big\langle h(s), \iint_{\mathbb{R}^{2n}} \overline{g(s-u)} e^{-2\pi i(-\nu)\cdot u} e^{-2\pi i(s+x)\cdot\xi}\varphi(u,\xi)\,du\,d\xi\Big\rangle \tag{4.8}$$

or as

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} e^{-2\pi i[(-\nu)\cdot u + x\cdot\xi]}\big\langle h(s), \overline{g(s-u)} e^{-2\pi i s\cdot\xi}\big\rangle \varphi(u,\xi)\,du\,d\xi \\
&\quad = \mathcal{F}\big\{(\mathcal{V}_g h)\varphi\big\}(-\nu, x).
\end{aligned} \tag{4.9}$$

To justify the equality between (4.8) and (4.9), consider a sequence (h_k) in $\mathcal{S}(\mathbb{R}^n)$ which converges to h in $\mathcal{S}'(\mathbb{R}^n)$ as $k\to\infty$. For fixed x and ν, let

$$\zeta(s) = \iint_{\mathbb{R}^{2n}} \overline{g(s-u)} e^{-2\pi i(-\nu)\cdot u} e^{-2\pi i(s+x)\cdot\xi}\varphi(u,\xi)\,du\,d\xi.$$

Since $\zeta\in\mathcal{S}(\mathbb{R}^n)$, we have $\langle h_k(s), \zeta(s)\rangle \to \langle h(s), \zeta(s)\rangle$ as $k\to\infty$. Furthermore, since, for each k,

$$\iiint_{\mathbb{R}^{3n}} \big|h_k(s)\big|\big|g(s-u)\big|\big|\varphi(u,\xi)\big|\,du\,d\xi\,ds < \infty,$$

Fubini's theorem shows that $\langle h_k(s), \zeta(s)\rangle = \mathcal{F}\{(\mathcal{V}_g h_k)\varphi\}(-\nu, x)$. Since the sequence $\{h_k\}$ is convergent in $\mathcal{S}'(\mathbb{R}^n)$, it must be weakly bounded and thus strongly bounded in $\mathcal{S}'(\mathbb{R}^n)$. Using Lemma 4.3 with $\alpha=\beta=0$, we can find a constant C and an integer m, such that, for all k,

$$\big|(\mathcal{V}_g h_k)(x,\nu)\big| \leqslant C\big(1+|x|^2+|\nu|^2\big)^m, \quad (x,\nu)\in\mathbb{R}^{2n},$$

and thus

$$\big|(\mathcal{V}_g h_k)(x,\nu)\varphi(x,\nu)\big| \leqslant \frac{C'}{(1+|x|^2+|\nu|^2)^{n+1}}, \quad (x,\nu)\in\mathbb{R}^{2n},$$

for some constant C' independent of k. Since the right-hand side of the previous inequality is integrable on $\mathbb{R}^{2n}$ and $\mathcal{V}_g h_k$ converges to $\mathcal{V}_g h$ pointwise as $k\to\infty$, our claim follows from the Lebesgue dominated convergence theorem. Noting that, by Lemma 4.3, the function $(\mathcal{V}_g h)\varphi$ belongs to $\mathcal{S}(\mathbb{R}^{2n})$ and letting $S=\Phi(T)$, we have thus

$$\langle \mathcal{F}_2 S, \varphi\rangle = \int_{\mathbb{R}^{2n}} \mathcal{F}\{(\mathcal{V}_g h)\varphi\}(-\nu, x)\, d\mu(x,\nu) = \langle \mu(x,\nu), \mathcal{F}\{(\mathcal{V}_g h)\varphi\}(-\nu, x)\rangle$$
$$= \langle \mathcal{F}^S \mu(x,\nu), (\mathcal{V}_g h)(x,\nu)\varphi(x,\nu)\rangle = \langle \mathcal{F}^S \mu(x,\nu)(\mathcal{V}_g h)(x,\nu), \varphi(x,\nu)\rangle$$

while $\mathcal{F}_2 \Phi(\rho) = \delta_{(0,0)}(x,\nu)$, from which the identity (4.7) follows immediately. Conversely, if the identity (4.7) holds, the preceding arguments can easily be reversed to obtain (4.6), which concludes the proof. $\square$

Note that the identity (4.6) means that every function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ admits the expansion

$$\varphi(t) = \int_{\mathbb{R}^{2n}} \mathcal{V}_g \varphi(x,\nu) e^{2\pi i t\cdot\nu} h(t-x)\, d\mu(x,\nu)$$

in the space $\mathcal{S}'(\mathbb{R}^n)$.

Remark. The product appearing on the left-hand side formula (4.7) is well defined in $\mathcal{S}'(\mathbb{R}^{2n})$ as the product between a distribution in $\mathcal{S}'(\mathbb{R}^{2n})$ and a function in $\mathcal{O}_M(\mathbb{R}^{2n})$. By taking the inverse Fourier transform of both sides of (4.7) and using the fact that the so-called cross Rihaczek distribution of g and h, defined by

$$\mathcal{R}(g,h)(x,\nu) = \left(g(x) \otimes \overline{\hat{h}(\nu)}\right) e^{-2\pi i x\cdot\nu}, \quad (x,\nu) \in \mathbb{R}^{2n},$$

satisfies the identity (see [10, Lemma 8.9] for more details)

$$\mathcal{F}\{\mathcal{R}(g,h)\}(u,\xi) = \mathcal{V}_g h(-\xi, u), \quad (u,\xi) \in \mathbb{R}^{2n},$$

we deduce that (4.7) is equivalent to

$$\mu * \mathcal{R}(g,h) = 1 \quad \text{on } \mathbb{R}^{2n}.$$

The following lemma is needed before stating the L^2-version of the previous theorem.

Lemma 4.5. *Let μ be a complex Borel measure on $\mathbb{R}^m$ with the property that its total variation, $|\mu|$, is translation-bounded. Suppose that for some $r > 0$ and some $\tau \in \mathbb{R}^m$,*

$$\operatorname{supp}(\mathcal{F}\mu) \cap B_r(\tau) = \{\tau\},$$

where $B_r(\tau) = \{\xi \in \mathbb{R}^m, |\xi - \tau| < r\}$. Then, there exists $a \in \mathbb{C}$ such that

$$\mathcal{F}\mu = a\delta_\tau \quad \text{on } B_r(\tau).$$

Proof. Choose a function φ which is infinitely differentiable with compact support in the ball $B = \{\xi \in \mathbb{R}^m, |\xi| < 1\}$, and define

$$\psi_\epsilon(\xi) = \varphi\left(\frac{\xi - \tau}{\epsilon}\right), \quad \text{for } 0 < \epsilon < r.$$

Then, $\mathcal{F}\psi_\epsilon(x) = e^{-2\pi i x\cdot\tau}\epsilon^m\mathcal{F}\varphi(\epsilon x)$ and, thus,

$$\bigl|\langle\mu,\mathcal{F}\psi_\epsilon\rangle\bigr| = \biggl|\int_{\mathbb{R}^m} e^{-2\pi i x\cdot\tau}\epsilon^m\mathcal{F}\varphi(\epsilon x)\,d\mu(x)\biggr| \leqslant \epsilon^m\int_{\mathbb{R}^m}\bigl|\mathcal{F}\varphi(\epsilon x)\bigr|\,d|\mu|(x).$$

Since $\mathcal{F}\varphi\in\mathcal{S}(\mathbb{R}^m)$, there exists, for each integer $N\geqslant 1$, a constant B_N such that

$$\bigl|\mathcal{F}\varphi(x)\bigr| \leqslant \frac{B_N}{1+|x|^{2N}}$$

and, since $|\mu|$ is translation-bounded, there exists $C>0$ such that

$$|\mu|\bigl([0,1)^m + x\bigr) \leqslant C,\quad \forall x\in\mathbb{R}^m.$$

Hence

$$\bigl|\langle\mu,\mathcal{F}\psi_\epsilon\rangle\bigr| \leqslant CB_N\epsilon^m\sum_{k\in\mathbb{Z}^m}\sup_{y\in[0,1)^m}\frac{1}{1+\epsilon^{2N}|k+y|^{2N}}. \tag{4.10}$$

Now, if $|k|\geqslant 6\sqrt{m}$ and $y_1, y_2\in[0,1)^m$, we have

$$|k+y_1|^2 \leqslant \bigl(|k|+|y_1|\bigr)^2 \leqslant \bigl(|k|+\sqrt{m}\bigr)^2 \leqslant 2\bigl(|k|-\sqrt{m}\bigr)^2 \leqslant 2|k+y_2|^2.$$

This shows that, if $|k|\geqslant 6\sqrt{m}$,

$$\sup_{y\in[0,1)^m}\frac{1}{1+\epsilon^{2N}|k+y|^{2N}} \leqslant \int_{[0,1)^m}\frac{1}{1+\epsilon^{2N}2^{-2N}|k+y|^{2N}}\,dy.$$

Therefore, using this last estimate together with (4.10), we obtain, letting A be the cardinality of the set $\{k\in\mathbb{Z}^m, |k|<6\sqrt{m}\}$, that

$$\begin{aligned}\bigl|\langle\mu,\mathcal{F}\psi_\epsilon\rangle\bigr| &\leqslant CB_N\epsilon^m\Biggl\{A + \sum_{\substack{k\in\mathbb{Z}^m\\ |k|\geqslant 6\sqrt{m}}}\int_{[0,1)^m}\frac{1}{1+\epsilon^{2N}2^{-2N}|k+y|^{2N}}\,dy\Biggr\}\\ &\leqslant CB_N\epsilon^m\biggl\{A + \int_{\mathbb{R}^m}\frac{1}{1+|\epsilon x/2|^{2N}}\,dx\biggr\}\\ &= CB_N\epsilon^m\biggl\{A + \epsilon^{-m}2^m\int_{\mathbb{R}^m}\frac{1}{1+|x|^{2N}}\,dx\biggr\}.\end{aligned}$$

If $N>m/2$, it follows that the integral above is finite and we obtain the existence of a constant $M>0$ such that

$$\bigl|\langle\mu,\mathcal{F}\psi_\epsilon\rangle\bigr| \leqslant M,\quad 0<\epsilon<r.$$

On the other hand, our assumption on the support of $\mathcal{F}\mu$ implies that

$$\mathcal{F}\mu = \sum_{\alpha \in F} c_\alpha D^\alpha \delta_\tau \quad \text{on } B_r(\tau),$$

where $F \subset \mathbb{N}^m$ is a finite set of multi-indices, by a well-known characterization of distributions supported on a single point (see [24]). We have thus, if $0 < \epsilon < r$,

$$\left|\langle \mu, \mathcal{F}\psi_\epsilon \rangle\right| = \left|\langle \mathcal{F}\mu, \psi_\epsilon \rangle\right| = \left| \sum_{\alpha \in F} c_\alpha (-1)^{|\alpha|} \epsilon^{-|\alpha|} D^\alpha \varphi(0) \right|.$$

Since the values of the partial derivatives of φ at 0 can be arbitrary, this last expression will not remain bounded for all $\varphi \in \mathcal{C}_0^\infty(B_r)$ as ϵ approaches 0 unless $c_\alpha = 0$ whenever $\alpha \neq 0$. This proves our claim. □

Theorem 4.6. *Let μ be a translation-bounded, positive Borel measure on $\mathbb{R}^{2n}$. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and let $h \in L^2(\mathbb{R}^n)$ satisfies the Bessel condition* (1.6) (*with g replaced with h*) *with respect to μ. Then, the function h is a dual window for the pair (g, μ) if and only if*

$$\left(\mathcal{F}^S \mu\right)(\mathcal{V}_g h) = \delta_{(0,0)} \quad \textit{on } \mathbb{R}^{2n}. \tag{4.11}$$

In particular, if h is a dual window for the pair (g, μ), we must have that $(g, h)_2 \neq 0$.

Proof. If h is a dual window for the pair (g, μ), the identity (4.1) has to hold, in particular, for all functions in $\mathcal{S}(\mathbb{R}^n)$. This implies (4.11) using Theorem 4.4. Conversely, using that same result, if (4.11) holds, the identity (4.6) holds for all functions in $\mathcal{S}(\mathbb{R}^n)$. Since h satisfies (1.6) by assumption and g as well by Proposition 2.4, it follows that the identity (4.6) can be extended to all functions in $L^2(\mathbb{R}^n)$ by continuity, showing that h is a dual window for the pair (g, μ). Finally, if h is a dual window for the pair (g, μ), the identity (4.7) implies that the support of $\mathcal{F}^S \mu$ is the set $\{(0,0)\}$ on a neighborhood of the origin. Using Lemma 4.5, it follows that $\mathcal{F}^S \mu = C\delta_{0,0}$ on that neighborhood. Thus $C\mathcal{V}_g h(0,0) = C(g,h)_2 = 1$, showing that $(g,h)_2 \neq 0$. □

Note that the well-known orthogonality conditions for the short-time Fourier transform (see [9, Theorem 3.2.1]) can be obtained from the previous theorem in the case where one of the windows belongs to the Schwartz class, since, under the condition that $(g,h)_2 = 1$, the identity (4.11) holds when $\mu = 1$. We now consider the analogue of Corollary 3.3 for Gabor dual systems.

Corollary 4.7. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$ satisfy $(g,h)_2 = 1$. Then, the measure $\mu = 1$ is the only translation-bounded positive Borel measure on $\mathbb{R}^{2n}$ such that h is a dual window for the pair (g, μ) if and only if*

$$\left|\mathcal{V}_g h(x,\nu)\right|^2 + \left|\mathcal{V}_g h(-x,-\nu)\right|^2 \neq 0, \quad \textit{for all } (x,\nu) \in \mathbb{R}^{2n}. \tag{4.12}$$

Proof. If the condition (4.12) holds and h is a dual window for the pair (g, μ), we must have

$$\mathcal{F}^S(\mu - 1)(x,\nu)\mathcal{V}_g h(x,\nu) = 0$$

which implies that

$$\mathcal{F}^S(\mu - 1)(x, \nu)\left|\mathcal{V}_g h(x, \nu)\right|^2 = 0.$$

Since $\mu - 1$ is a real measure, we have $\overline{\mathcal{F}^S(\mu - 1)(x, \nu)} = \mathcal{F}^S(\mu - 1)(-x, -\nu)$. This together with the previous identity yields

$$\mathcal{F}^S(\mu - 1)(x, \nu)\left|\mathcal{V}_g h(-x, -\nu)\right|^2 = 0$$

and thus

$$\mathcal{F}^S(\mu - 1)(x, \nu)\left\{\left|\mathcal{V}_g h(x, \nu)\right|^2 + \left|\mathcal{V}_g h(-x, -\nu)\right|^2\right\} = 0.$$

Hence, $\mathcal{F}^S(\mu - 1) = 0$ and $\mu = 1$. If the condition (4.12) is not satisfied at some point (x_0, ν_0), the construction of a measure $\mu \neq 1$ such that h is a dual window for the pair (g, μ) is the same as in Corollary 3.3. Note that, since the measure μ constructed is given by a function in $L^\infty(\mathbb{R}^{2n})$, any function in $L^2(\mathbb{R}^n)$ satisfies the Bessel condition (1.6) with respect to μ as can be seen using Theorem 1.1. □

Example. Consider the pair of windows $g(t) = e^{-\pi t^2}$ and $h(t) = -\frac{\sqrt{2}}{\zeta}(t - \zeta)e^{-\pi t^2}$, where $\zeta \in \mathbb{C} \setminus \{0\}$, in one dimension. Then,

$$\mathcal{V}_g h(x, \nu) = -\frac{1}{2\zeta}(x - i\nu - 2\zeta)e^{-\pi(x^2+\nu^2+2ix\nu)/2}, \quad (x, \nu) \in \mathbb{R}^2.$$

Since, $\mathcal{V}_g h$ has a single zero located at the point $(x, \nu) = (2\operatorname{Re}(\zeta), -2\operatorname{Im}(\zeta))$, the condition (4.12) in the previous corollary is satisfied and, since $(g, h)_2 = 1$, the only translation-bounded positive Borel measure μ on $\mathbb{R}^{2n}$ such that h is a dual window for the pair (g, μ) is the trivial one, $\mu = 1$.

We also have the analogue of Proposition 3.14 for dual windows, with a similar proof which uses the identity (4.7).

Proposition 4.8. *Let μ be a positive tempered measure on $\mathbb{R}^{2n}$ and assume that $g_0 \in \mathcal{S}(\mathbb{R}^n)$ and $h_0 \in \mathcal{S}'(\mathbb{R}^n)$ satisfy $\langle h_0, \overline{g_0}\rangle = 1$ as well as* (4.6) *with $g = g_0$ and $h = h_0$. Then, there exists $r > 0$, such that*

$$\mathcal{F}\mu = \delta_{(0,0)} \quad \text{on } B_r, \tag{4.13}$$

where B_r denotes the open ball of radius r centered at the origin in $\mathbb{R}^{2n}$. In particular, if K is a compact subset of $\mathbb{R}^{2n}$ and ρ is tempered measure on $\mathbb{R}^{2n}$ different from μ such that $\mu = \rho$ on $\mathbb{R}^{2n} \setminus K$, then ρ fails to satisfy the identity (4.6) *for any window $g \in \mathcal{S}(\mathbb{R}^n)$ and any distribution $h \in \mathcal{S}'(\mathbb{R}^n)$.*

Note that the previous result has the following somewhat surprising consequence. Suppose for example that the collection $\{e^{2\pi ibl\cdot t}g(t - ak)\}_{(k,l)\in\mathbb{Z}^n\times\mathbb{Z}^n}$, where $a, b > 0$ are two parameters and $g \in \mathcal{S}(\mathbb{R}^n)$ forms a frame for $L^2(\mathbb{R}^n)$. Then, the standard dual of the frame is again a Gabor system $\{e^{2\pi i bl\cdot t}h(t-ak)\}_{(k,l)\in\mathbb{Z}^n\times\mathbb{Z}^n}$ with $h\in L^2(\mathbb{R}^n)$. However, if a small non-zero perturbation is applied to the frame as in Proposition 4.8, the resulting system will still be a frame but it can no longer admit a dual which is of "Gabor type."

We will now use Theorem 4.6 to obtain more information about the measures μ having the property that the pair (φ,μ) admits a dual window in the case where φ is the Gaussian $\varphi(x)=2^{n/4}e^{-\pi|x|^2}$. Recall that the Bargmann–Fock space $\mathcal{F}^2(\mathbb{C}^n)$ is the Hilbert space of entire functions F on $\mathbb{C}^n$ for which

$$\|F\|_{\mathcal{F}}^2=\int_{\mathbb{C}^n}\bigl|F(z)\bigr|^2e^{-\pi|z|^2}\,dz$$

is finite. It is known that the Bargmann transform $B:L^2(\mathbb{R}^n)\to\mathcal{F}^2(\mathbb{C}^n)$, defined by

$$Bf(z)=2^{n/4}\int_{\mathbb{R}^n}f(t)e^{2\pi t\cdot z-\pi t^2-\frac{\pi}{2}z^2}\,dt,\quad f\in L^2\bigl(\mathbb{R}^n\bigr),\ z\in\mathbb{C}^n,$$

is an isometry from $L^2(\mathbb{R}^n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$. Furthermore, if we write $z=x+i\nu\in\mathbb{C}^n$, then we have

$$\mathcal{V}_\varphi f(x,-\nu)=e^{\pi i x\cdot\nu}e^{-\pi|z|^2/2}Bf(z),\quad f\in L^2\bigl(\mathbb{R}^n\bigr).$$

(See [9, Section 3.4] for more details on the Bargmann transform.) It was proved by Lyubarskiĭ [20] and, independently by Seip and Wallstén [25,26], that, in dimension $n=1$, the regular Gabor system generated by the Gaussian $\{e^{2\pi i blt}\varphi(t-ak)\}_{(k,l)\in\mathbb{Z}^2}$, where $a,b>0$ are two parameters, forms a frame for $L^2(\mathbb{R})$ if $ab<1$. In particular, in that case, the standard dual provides a dual of Gabor type for the system. Note that this situation corresponds to the case where the measure μ associated with the system is the counting measure of a lattice and its symplectic Fourier transform, $\mathcal{F}^S\mu$, is thus a measure supported on the adjoint lattice (see the discussion at the beginning of Section 5) and it is in particular discrete. The following result, which is only valid in dimension $n=1$, shows, in particular, that, if μ is any measure such that the pair (φ,μ) admits a dual window $h\in L^2(\mathbb{R})$, then the support of $\mathcal{F}^S\mu$ must necessarily be discrete.

Proposition 4.9. *Let $\varphi(x)=2^{1/4}e^{-\pi x^2}$ and let μ be a translation-bounded measure on $\mathbb{R}^2$. Suppose that the pair (φ,μ) admits a dual window $h\in L^2(\mathbb{R})$ with $(\varphi,h)_2=1$, Then, there exists a discrete set $\Lambda\subset\mathbb{R}^2$ with $-\Lambda=\Lambda$ and a function $F\in\mathcal{F}^2(\mathbb{C})$ satisfying $F(0)=1$ such that*

$$\mathcal{F}^S\mu=\delta_{(0,0)}+\sum_{(x,\nu)\in\Lambda}c_{x,\nu}\delta_{(x,\nu)}\tag{4.14}$$

where

$$\Lambda\subset Z_F:=\bigl\{(x,\nu)\in\mathbb{R}^2,\ F(x-i\nu)=F(-x+i\nu)=0\bigr\}$$

and $c_{x,\nu}\in\mathbb{C}\setminus\{0\}$ for each $(x,\nu)\in\Lambda$.

Conversely, given any function $F \in \mathcal{F}^2(\mathbb{C})$ *with* $F(0) = 1$ *and any subset* Λ *of* $\mathbb{R}^2$ *satisfying* $-\Lambda = \Lambda$ *and* $\Lambda \subset Z_F$*, there exists a translation-bounded measure* μ *on* $\mathbb{R}^2$ *such that* $\mathcal{F}^S\mu$ *is of the form* (4.14) *and a function* $h \in L^2(\mathbb{R})$ *such that the pair* (φ, μ) *admits* h *as a dual window.*

Proof. Suppose first that $h \in L^2(\mathbb{R})$ is a dual window for the pair (φ, μ). Since μ is a positive measure, we have $\overline{\mathcal{F}^S\mu(x, \nu)} = \mathcal{F}^S\mu(-x, -\nu)$. Hence, if $\mathcal{F}^S\mu$ has the form (4.14), we must have $-\Lambda = \Lambda$ (and $c_{-x,-\nu} = \overline{c_{x,\nu}}$ for $(x, \nu) \in \Lambda$). To prove that (4.14) holds, we use Theorem 4.6 to obtain the identity

$$\big(\mathcal{F}^S\mu\big)(x, \nu)\mathcal{V}_\varphi h(x, \nu) = \delta_{(0,0)}(x, \nu)$$

which is equivalent to

$$\big(\mathcal{F}^S\mu\big)(x, \nu)Bh(x - i\nu) = \delta_{(0,0)}(x, \nu).$$

This last identity combined with the fact that the support of $\mathcal{F}^S$ is invariant under the transformation $(x, \nu) \mapsto (-x, -\nu)$ implies that the support of $\mathcal{F}^S\mu$ is contained in the union of the set $\{(0, 0)\}$ with Z_F, where $F := Bh \in \mathcal{F}^2(\mathbb{C})$. Since F is an entire function of one complex variable, its zeros are isolated and the representation (4.14) follows from Lemma 4.5. Conversely, if $F \in \mathcal{F}^2(\mathbb{C})$ satisfies $F(0) = 1$ and Λ is a subset of Z_F with $-\Lambda = \Lambda$, let $L = \{(x_k, \nu_k), k \in K\}$ be a subset of Z_F such that $\Lambda = L \cup (-L)$ and $L \cap (-L) = \emptyset$ where $K = \emptyset$ if $\Lambda = \emptyset$, $K = \{1, 2, \dots, m\}$ if $\operatorname{card}(L) = m$ and $K = \mathbb{N} = \{1, 2, \dots\}$ if L is infinite. Defining μ by letting

$$\mathcal{F}^S\mu = \delta_{(0,0)} + \sum_{k\in K} 4^{-k}\big(\delta_{(x_k,\nu_k)} + \delta_{(-x_k,-\nu_k)}\big)$$

we have $\mu = g \in L^\infty(\mathbb{R}^2)$ with $1/3 \leqslant g \leqslant 5/3$ and the result follows immediately from Theorem 4.6 letting $h = B^{-1}F$. □

Remark. It is clear that in dimension $n \geqslant 2$, the proposition just proved is not true. For example, if $h \in L^2(\mathbb{R}^2)$ is of the form $h = h_1 \otimes h_2$ where $h_1, h_2 \in L^2(\mathbb{R})$ and $\varphi = \varphi_1 \otimes \varphi_1$, where $\varphi_1(x) = 2^{1/4}e^{-\pi|x|^2}$, we have

$$\mathcal{V}_\varphi h(x_1, x_2, \nu_1, \nu_2) = \mathcal{V}_{\varphi_1} h_1(x_1, \nu_1)\mathcal{V}_{\varphi_1} h_1(x_2, \nu_2)$$

and if, for example, $\mathcal{V}_{\varphi_1} h_1(x_0, \nu_0) = \mathcal{V}_{\varphi_1} h_1(-x_0, -\nu_0) = 0$, then

$$\mathcal{V}_\varphi h(x_0, x_2, \nu_0, \nu_2) = \mathcal{V}_\varphi h(-x_0, -x_2, -\nu_0, -\nu_2) = 0$$

for all $(x_2, \nu_2) \in \mathbb{R}^2$ and the zeros of $\mathcal{V}_\varphi h$ are not isolated. It is then easy to use Theorem 4.6 to construct a counterexample.

Definition 4.10. The *cross-ambiguity function* of two functions f and g in $L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{A}(f, g)(x, \nu) = \int_{\mathbb{R}^n} f(t + x/2)\overline{g(t - x/2)}e^{-2\pi i\nu\cdot t}\,dt, \quad (x, \nu) \in \mathbb{R}^{2n}. \tag{4.15}$$

Lemma 4.11. *Let g and h be two generalized Gaussian on $\mathbb{R}^n$ of the form*

$$g(x) = e^{-\pi Ax\cdot x+\pi a\cdot x}, \qquad h(x) = e^{-\pi Bx\cdot x+\pi b\cdot x}$$

where both A and B are symmetric matrices in $GL(n,\mathbb{C})$ with positive definite real part and $a, b \in \mathbb{C}^n$. Then,

$$\mathcal{A}(g,h)(x,\nu) = e^{R(x,\nu)}, \quad (x,\nu) \in \mathbb{R}^{2n},$$

where $R(x,\nu)$ is a polynomial of degree 2 (in $2n$ variables).

Proof. First note that, if A satisfies the conditions of the lemma, then

$$\mathcal{F}\{e^{-\pi Ax\cdot x}\}(\xi) = \int_{\mathbb{R}^n} e^{-\pi Ax\cdot x} e^{-2\pi i\xi\cdot x}\,dx = (\det A)^{-1/2} e^{-\pi A^{-1}\xi\cdot\xi}, \quad \xi \in \mathbb{R}^n.$$

In the previous formula, the term $(\det A)^{1/2}$ is well defined (and positive) if A is positive definite and can be extended by analyticity to the set of symmetric matrices with positive definite real part, since the range of the mapping $A \mapsto \det A$, where A varies over such matrices, is a simply connected open subset of $\mathbb{C} \setminus \{0\}$. (See [9,13] for more details.) Since the integral defining the Fourier transform is absolutely convergent even when $\xi \in \mathbb{C}^n$ and the terms in the previous identity are all well defined and analytic when viewed as functions of $\xi \in \mathbb{C}^n$, it follows, by analyticity, that the previous identity also holds for $\xi \in \mathbb{C}^n$. Therefore, we have

$$\begin{aligned}
\mathcal{F}\{e^{-\pi Ax\cdot x+2\pi a\cdot x}\}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi Ax\cdot x+2\pi a\cdot x} e^{-2\pi i\xi\cdot x}\,dx \\
&= \int_{\mathbb{R}^n} e^{-\pi Ax\cdot x} e^{-2\pi i(\xi+ia)\cdot x}\,dx \\
&= (\det A)^{-1/2} e^{-\pi A^{-1}(\xi+ia)\cdot(\xi+ia)}, \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathcal{A}(g,h)(x,\nu) \\
&\quad = \int_{\mathbb{R}^n} e^{-\pi A(t+x/2)\cdot(t+x/2)+2\pi a\cdot(t+x/2)} e^{-\pi B^*(t-x/2)\cdot(t-x/2)+2\pi\bar{b}\cdot(t-x/2)} e^{-2\pi i\nu\cdot x}\,dt \\
&\quad = e^{-\pi(A+B^*)x\cdot x/4+\pi(a-\bar{b})\cdot x} \int_{\mathbb{R}^n} e^{-\pi(A+B^*)t\cdot t+2\pi(a-\bar{b})\cdot t} e^{-\pi(A-B^*)x\cdot t} e^{-2\pi i\nu\cdot t}\,dt \\
&\quad = e^{-\pi(A+B^*)x\cdot x/4+\pi(a-\bar{b})\cdot x} (\det C)^{-1/2} e^{-\pi C^{-1}\xi\cdot\xi},
\end{aligned}$$

where $C = A + B^*$ and $\xi = \nu + i(a - \bar{b} + (B^* - A)x/2)$. The conclusion of the lemma follows. $\square$

As an immediate consequence, we have the following result.

Theorem 4.12. *Let $g(x) = c_1 e^{-\pi Ax\cdot x+\pi a\cdot x}$ and $h(x) = c_2 e^{-\pi Bx\cdot x+\pi b\cdot x}$, where both A and B are symmetric matrices in $GL(n,\mathbb{C})$ with positive definite real part, $a, b \in \mathbb{C}^n$ and c_1, c_2 are constants chosen so that $(g,h)_2 = 1$. Then, the measure $\mu = 1$ is the only positive tempered measure on $\mathbb{R}^{2n}$ such that the identity of* (4.6) *holds for this choice of g and h. In particular, the measure $\mu = 1$ is the only positive, translation-bounded measure on $\mathbb{R}^{2n}$ such that h is a dual for the pair (g,μ).*

Proof. The result follows immediately from Lemma 4.11 and Theorems 4.4 and 4.6, using the identity (3.6). □

If the generalized Gaussians in the previous theorem are multiplied by polynomials, the uniqueness of the measure μ is no longer true, but the impossibility to construct discrete measure solution of the problem still remains as our next result will show. Before stating it, we need first the following lemma whose proof is almost obvious in the case where the set Λ supporting the measure μ is not dense in $\mathbb{R}^m$. A bit more work is needed if we do not make this assumption.

Lemma 4.13. *Let $\mu = \sum_{a\in\Lambda} c_a\delta_a \in \mathcal{S}'(\mathbb{R}^m)$ be a discrete measure on $\mathbb{R}^m$, where $\Lambda \subset \mathbb{R}^m$ is at most countable, and assume that, for some $N > 0$,*

$$\int_{\mathbb{R}^m} \frac{1}{(1+|x|^2)^N}\, d|\mu| < \infty,$$

where $|\mu|$ denotes the total variation of μ, so that, in particular, μ is a tempered measure on $\mathbb{R}^m$. Then, there exists no polynomial of m variable $P(\xi)$ such that

$$\mathcal{F}\mu(\xi)P(\xi) = \delta_0 \quad on\ \mathbb{R}^m. \tag{4.16}$$

Proof. The inverse Fourier transform of $P(\xi)$ has the form

$$\mathcal{F}^{-1}\big(P(\xi)\big) = \sum_{\alpha\in F} b_\alpha D^\alpha \delta_0$$

where F is a finite subset of $\mathbb{N}^m$ and $b_\alpha \neq 0$ for each $\alpha \in F$. Taking inverse Fourier transform, Eq. (4.16) becomes

$$\mu * \sum_{\alpha\in F} b_\alpha D^\alpha \delta_0 = 1. \tag{4.17}$$

Choose a function φ which is infinitely differentiable and also compactly supported in the ball $\{x \in \mathbb{R}^m, |x| < 1\}$ and satisfies $\varphi(0) = 1$. Choose $\gamma \in F$ such that $|\alpha| \leqslant |\gamma|$ for all $\alpha \in F$. Fix $d \in \Lambda$ with $\mu(\{d\}) = c_d \neq 0$ and define

$$\psi_\epsilon(x) = (x-d)^\gamma \varphi\left(\frac{x-d}{\epsilon}\right), \quad \text{for } \epsilon > 0.$$

We have

$$\begin{aligned}\{D^\gamma\psi_\epsilon\}(x) &= \gamma!\varphi\left(\frac{x-d}{\epsilon}\right)\\&\quad+\sum_{0\leqslant\beta<\gamma}\binom{\gamma}{\beta}\frac{\gamma!}{(\gamma-\beta)!}\left(\frac{x-d}{\epsilon}\right)^{\gamma-\beta}\left(D^{\gamma-\beta}\varphi\right)\left(\frac{x-d}{\epsilon}\right)\\&= \gamma!\varphi\left(\frac{x-d}{\epsilon}\right)+\zeta\left(\frac{x-d}{\epsilon}\right)\end{aligned}\tag{4.18}$$

where $\zeta\in C_0^\infty(\mathbb{R}^m)$ is supported in the ball $\{x\in\mathbb{R}^m, |x|<1\}$ and satisfies $\zeta(0)=0$. Similarly, if $\alpha\in F$ and $\alpha\neq\gamma$, we have

$$\begin{aligned}\{D^\alpha\psi_\epsilon\}(x) &= \sum_{\substack{\beta\leqslant\gamma\\\beta\leqslant\alpha}}\binom{\alpha}{\beta}\frac{\gamma!}{(\gamma-\beta)!}(x-d)^{\gamma-\beta}\frac{1}{\epsilon^{|\alpha|-|\beta|}}(D^{\alpha-\beta}\varphi)\left(\frac{x-d}{\epsilon}\right)\\&= \sum_{\substack{\beta\leqslant\gamma\\\beta\leqslant\alpha}}\binom{\alpha}{\beta}\frac{\gamma!}{(\gamma-\beta)!}\left(\frac{x-d}{\epsilon}\right)^{\gamma-\beta}\epsilon^{|\gamma|-|\alpha|}\left(D^{\alpha-\beta}\varphi\right)\left(\frac{x-d}{\epsilon}\right).\end{aligned}$$

This shows that

$$\{D^\alpha\psi_\epsilon\}(x)=\sum_{k=0}^{|\alpha|}\epsilon^k\rho_k\left(\frac{x-d}{\epsilon}\right),\tag{4.19}$$

where, for each k, $\rho_k\in C_0^\infty(\mathbb{R}^m)$ is supported in the ball $\{x\in\mathbb{R}^m, |x|<1\}$ and satisfies $\rho_k(0)=0$. Clearly,

$$\lim_{\epsilon\to0^+}\langle 1,\psi_\epsilon\rangle=\lim_{\epsilon\to0^+}\int_{\mathbb{R}^m}\psi_\epsilon(x)\,dx=0.\tag{4.20}$$

Also, letting $\Lambda_0=\Lambda\cap B(d,1)$, we have

$$\sum_{\alpha\in\Lambda_0}|c_\alpha|<\infty$$

and, if $0<\epsilon<1$,

$$\begin{aligned}&\left\langle\sum_{\alpha\in F}b_\alpha D^\alpha\mu,\psi_\epsilon\right\rangle\\&\quad=\sum_{\alpha\in F}b_\alpha\sum_{a\in\Lambda}c_a(-1)^\alpha D^\alpha\psi_\epsilon(a)\\&\quad=\sum_{\alpha\in F}b_\alpha c_d(-1)^\alpha D^\alpha\psi_\epsilon(d)+\sum_{\alpha\in F}b_\alpha\sum_{a\in\Lambda_0\setminus\{d\}}c_a(-1)^\alpha D^\alpha\psi_\epsilon(a).\end{aligned}$$

Using (4.18) and (4.19), we see that $D^\gamma \psi_\epsilon(d) = \gamma!$ and, as $\epsilon \to 0^+$, $D^\alpha \psi_\epsilon(d)$ approaches 0 if $\alpha \neq \gamma$. Furthermore, for any $\alpha \in F$,

$$\lim_{\epsilon \to 0^+} \sum_{a \in \Lambda_0 \setminus \{d\}} |c_a| \left| D^\alpha \psi_\epsilon(a) \right| = 0.$$

Therefore,

$$\lim_{\epsilon \to 0^+} \left\langle \sum_{\alpha \in F} b_\alpha D^\alpha \mu, \psi_\epsilon \right\rangle = b_\gamma c_d (-1)^\gamma \gamma! \neq 0. \tag{4.21}$$

The identity (4.17) is contradicted by (4.20) and (4.21), which proves the lemma. □

Theorem 4.14. *Let $g(x) = p(x)e^{-\pi Ax\cdot x + \pi a\cdot x}$ and $h(x) = q(x)e^{-\pi Bx\cdot x + \pi b\cdot x}$, where both A and B are symmetric matrices in $GL(n, \mathbb{C})$ with positive definite real part, $a, b \in \mathbb{C}^n$ and $p(x)$, $q(x)$ are polynomials in n variables chosen so that $(g, h)_2 \neq 0$. Let μ be a positive translation-bounded measure on $\mathbb{R}^{2n}$. Then, h cannot be a dual window for the pair (g, μ) if the measure μ is discrete. In particular, no weighted irregular tight Gabor frame can be constructed using g as a window function.*

Proof. Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$ and let $\alpha, \beta \in \mathbb{N}^n$ be multi-indices. We have

$$\begin{aligned}
&\mathcal{A}\big(t^\alpha \psi_1(t), t^\beta \psi_2(t)\big)(x, \nu) \\
&\quad = \int_{\mathbb{R}^n} (t + x/2)^\alpha \psi_1(t + x/2)(t - x/2)^\beta \overline{\psi_2(t - x/2)} e^{-2\pi i \nu\cdot t}\, dt \\
&\quad = \sum_{\substack{0 \leqslant \gamma \leqslant \alpha \\ 0 \leqslant \delta \leqslant \beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} (x/2)^{\alpha+\beta-\gamma-\delta} (-1)^{\beta-\delta} \int_{\mathbb{R}^n} t^{\gamma+\delta} \psi_1(t + x/2)\overline{\psi_2(t - x/2)} e^{-2\pi i \nu\cdot t}\, dt \\
&\quad = \sum_{\substack{0 \leqslant \gamma \leqslant \alpha \\ 0 \leqslant \delta \leqslant \beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} (x/2)^{\alpha+\beta-\gamma-\delta} \frac{(-1)^{\beta-\delta}}{(-2\pi i)^{|\gamma+\delta|}} \frac{\partial^{\gamma+\delta}}{\partial \nu^{\gamma+\delta}} \big\{\mathcal{A}(\psi_1, \psi_2)(x, \nu)\big\}.
\end{aligned}$$

Therefore, if g and h are as above with $p(x) = \sum_{\alpha \in F} c_\alpha x^\alpha$ and $q(x) = \sum_{\beta \in G} d_\beta x^\beta$, we have,

$$\mathcal{A}\big(g(t), h(t)\big)(x, \nu) = \sum_{\alpha \in F, \beta \in G} c_\alpha \overline{d_\beta} \mathcal{A}\big(t^\alpha e^{-\pi At\cdot t + \pi a\cdot t}, t^\beta e^{-\pi Bt\cdot t + \pi b\cdot t}\big)(x, \nu).$$

By the previous computations and Lemma 4.11, this last expression can be written in the form

$$\mathcal{A}\big(g(t), h(t)\big)(x, \nu) = R_1(x, \nu) e^{R_2(x,\nu)}$$

where R_1 and R_2 are both polynomials in $2n$ variables and with R_2 of degree at most 2. We can also assume that $R_2(0, 0) = 0$ (by replacing $R_2(x, \nu)$ by $R_2(x, \nu) - R_2(0, 0)$ if necessary). If μ

is a positive, translation-bounded measure on $\mathbb{R}^{2n}$ which satisfies the identity (4.1) with g and h as above, we have, by Theorem 4.6, that

$$\big(\mathcal{F}^S\mu\big)(x,\nu)R_1(x,\nu)e^{R_2(x,\nu)}=\delta_{(0,0)}(x,\nu)\quad \text{on } \mathbb{R}^{2n}$$

or, equivalently, multiplying both sides of the previous identity by $e^{-R_2(x,\nu)}$,

$$\big(\mathcal{F}^S\mu\big)(x,\nu)R_1(x,\nu)=\delta_{(0,0)}(x,\nu)\quad \text{on } \mathbb{R}^{2n}.$$

After a change of variables, we obtain that

$$(\mathcal{F}\mu)(x,\nu)R(x,\nu)=\delta_{(0,0)}(x,\nu)\quad \text{on } \mathbb{R}^{2n}$$

for the polynomial $R(x,\nu)=R_1(-\nu,x)$. Using Lemma 4.13, it follows that μ cannot be a discrete measure which proves our claim. □

We now prove the analogue of Theorem 4.12 for the family of extreme value windows $\psi_{k,m}(t)=e^{kt-me^t}$, $k,m>0$.

Proposition 4.15. *Let k_1,k_2,m_1,m_2 be positive numbers. Then, for $(x,\nu)\in\mathbb{R}^2$, we have*

$$\begin{aligned}&\mathcal{A}(\psi_{k_1,m_1},\psi_{k_2,m_2})(x,\nu)\\&\quad=e^{(k_1-k_2)x/2}\big[m_1e^{x/2}+m_2e^{-x/2}\big]^{-(k_1+k_2-2\pi i\nu)}\Gamma(k_1+k_2-2\pi i\nu).\end{aligned}$$

In particular,

(a) *$\mathcal{A}(\psi_{k_1,m_1},\psi_{k_2,m_2})$ does not vanish anywhere on $\mathbb{R}^2$.*
(b) *Let $g(t)=c_1\psi_{k_1,m_1}(t)$ and $h(t)=c_2\psi_{k_2,m_2}(t)$, where the constants c_1 and c_2 are chosen so that $(g,h)_2=1$. Then, the measure $\mu=1$ is the only positive tempered measure on $\mathbb{R}^2$ such that the identity of* (4.6) *holds for this choice of g and h. In particular, the measure $\mu=1$ is the only positive, translation-bounded measure on $\mathbb{R}^2$ such that h is a dual for the pair (g,μ).*

Proof. We have, for any $(x,\nu)\in\mathbb{R}^2$,

$$\begin{aligned}&\mathcal{A}(\psi_{k_1,m_1},\psi_{k_2,m_2})(x,\nu)\\&\quad=\int_{\mathbb{R}} e^{k_1(t+x/2)-m_1e^{t+x/2}}e^{k_2(t-x/2)-m_2e^{t-x/2}}e^{-2\pi i\nu t}\,dt\\&\quad=e^{(k_1-k_2)x/2}\int_{\mathbb{R}} e^{(k_1+k_2)t-(m_1e^{x/2}+m_2e^{-x/2})e^t}e^{-2\pi i\nu t}\,dt\\&\quad=e^{(k_1-k_2)x/2}\mathcal{F}\psi_{k_1+k_2,m_1e^{x/2}+m_2e^{-x/2}}(\nu)\\&\quad=e^{(k_1-k_2)x/2}\big[m_1e^{x/2}+m_2e^{-x/2}\big]^{-(k_1+k_2-2\pi i\nu)}\Gamma(k_1+k_2-2\pi i\nu),\end{aligned}$$

and the statement in (a) follows from the fact that the gamma function does not vanish anywhere while (b) is again a consequence of (a) together with Theorems 4.4 and 4.6. □

5. Regular Gabor systems

In this section, we briefly point out how some of the results obtained in the previous sections relate to known ones in the case of (unweighted) regular Gabor systems. Although these results are known, it is worthwhile to mention them as they are immediate consequences of our main theorems.

Let Λ be a $2n$-dimensional lattice in $\mathbb{R}^{2n}$. Then Λ can be described as the set $C\mathbb{Z}^{2n}$, for some $2n \times 2n$ invertible real matrix C. Consider the measure

$$\mu = \left|\det(C)\right| \sum_{k\in\mathbb{Z}^{2n}} \delta_{Ck}.$$

Then,

$$\mathcal{F}\mu = \sum_{k\in\mathbb{Z}^{2n}} \delta_{Dk},$$

where $D = (C^t)^{-1}$. Denoting by $\mathcal{J}$ the linear mapping from $\mathbb{R}^{2n}$ to itself defined by $\mathcal{J}(x, y) = (-y, x)$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have also

$$\mathcal{F}^S\mu = \sum_{k\in\mathbb{Z}^{2n}} \delta_{\mathcal{J}Dk}.$$

The lattice $\mathcal{J}D\mathbb{Z}^{2n}$ appearing in the previous formula is called the *adjoint lattice* and will be denoted by Λ°. Applying Theorem 3.2 to this particular translation-bounded measure μ, we obtain thus the following result.

Theorem 5.1. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ with $\|g\|_2 = 1$ and consider the lattice $\Lambda = C\mathbb{Z}^{2n}$, where C is a $2n \times 2n$ invertible real matrix and the adjoint lattice Λ°. Let $r = \sqrt{|\det(C)|}$. Then, the following are equivalent*:

(1) $\{re^{2\pi i\nu\cdot t} g(t - x)\}_{(x,\nu)\in\Lambda}$ *is a Parseval tight frame for* $L^2(\mathbb{R}^n)$.
(2) $\mathcal{V}_g g(x, \nu) = 0$, $(x, \nu) \in \Lambda^\circ \setminus \{(0, 0)\}$.
(3) *The collection* $\{e^{2\pi it\cdot\nu} g(t - x)\}_{(x,\nu)\in\Lambda^\circ}$ *is orthonormal.*

When restricted to the case of separable lattices, this result can be seen as a particular case of the Ron–Shen duality [22] or the Wexler–Raz identity [30] with the window being in the Schwartz class.

A similar result holds for dual systems, using Theorem 4.6.

Theorem 5.2. *Let Λ, Λ° and r be as in the previous theorem. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and let $h \in L^2(\mathbb{R}^n)$ be such that the collection $\{e^{2\pi i\nu\cdot t} h(t - x)\}_{(x,\nu)\in\Lambda}$ has the Bessel property and assume that $(g, h)_2 = 1$. Then, the following are equivalent*:

(1) *The collections* $\{re^{2\pi i\nu\cdot t}g(t-x)\}_{(x,\nu)\in\Lambda}$ *and* $\{re^{2\pi i\nu\cdot t}h(t-x)\}_{(x,\nu)\in\Lambda}$ *are Gabor duals in* $L^2(\mathbb{R}^n)$.
(2) $\mathcal{V}_g h(x,\nu)=0$, $(x,\nu)\in\Lambda^\circ\setminus\{(0,0)\}$.
(3) *The collections* $\{e^{2\pi it\cdot\nu}g(t-x)\}_{(x,\nu)\in\Lambda^\circ}$ *and* $\{e^{2\pi it\cdot\nu}h(t-x)\}_{(x,\nu)\in\Lambda^\circ}$ *are biorthogonal.*

Again, this last result, when restricted to the case of separable lattices, yields the Wexler–Raz identity [30], of course with the restriction that one of the windows belongs to the Schwartz class. These results were obtained in full generality for windows in $L^2(\mathbb{R}^n)$ in [6]. (See also [9, Section 9.4] for the case of symplectic lattices.)

6. Beurling density and Gabor duality

In this last section, we study the relationship between our characterization formula (4.11) for Gabor duality and the Beurling density of the corresponding measure μ. Several authors have obtained density results for discrete irregular Gabor systems showing that certain properties of the system such as being a frame or a Riesz basis have certain implications on the upper and lower Beurling densities associated with the corresponding sampling points in the time–frequency space (see [1,3,21] for the case of unweighted systems and [16] for weighted ones). We will define analogously the upper and lower Beurling density of a positive Borel measure and show that if a Gabor system associated with a measure admits a dual in the sense of Definition 4.1, then necessarily the associated measure has a well-defined Beurling density which must be equal to one. As we will see, this property is a direct consequence of the fact that the measure is translation-bounded and that its Fourier transform is equal to the Dirac mass at 0 in a neighborhood of the origin.

Given a point $z\in\mathbb{R}^m$ and $r>0$, we denote by $Q_r(z)$ the cube in $\mathbb{R}^m$ centered at z with side length r. Given a positive Borel measure μ on $\mathbb{R}^m$, we define its upper and lower Beurling density, $D^+(\mu)$ and $D^-(\mu)$, by the formulas

$$D^+(\mu)=\limsup_{r\to\infty}\sup_{z\in\mathbb{R}^m}\frac{\mu(Q_r(z))}{r^m},\qquad D^-(\mu)=\liminf_{r\to\infty}\inf_{z\in\mathbb{R}^m}\frac{\mu(Q_r(z))}{r^m},$$

and, if these two densities are equal, we define the Beurling density of μ to be $D(\mu)=D^+(\mu)=D^-(\mu)$.

Lemma 6.1. *Let* μ *be a positive translation-bounded Borel measure on* $\mathbb{R}^m$*. Then,*

(a) *There exists constant* C *such that*

$$\limsup_{r\to\infty}\sup_{z\in\mathbb{R}^m}\frac{\mu(Q_r(z))}{r^m}\leqslant C.$$

(b) *We have*

$$\lim_{r\to\infty}r^{-m}\left|\mu\big(Q_r(z)\big)-\mu\big(Q_r(z-u)\big)\right|=0,$$

uniformly for $z\in\mathbb{R}^m$ *and* $u\in Q_1(0)$.

Proof. To prove the statement in (a), note that, since μ is translation-bounded, there is a constant $C > 0$ such that $\sup_{y\in\mathbb{R}^m} \mu(Q_1(y)) \leqslant C$. Denoting by $[r]$ the integer part of r, we have the inclusion $Q_r(z) \subset Q_{[r]+1}(z)$ and this last set can be written as the union of $([r]+1)^m$ cubes of side length 1. Hence,

$$\begin{aligned}\limsup_{r\to\infty} \sup_{z\in\mathbb{R}^m} \frac{\mu(Q_r(z))}{r^m} &\leqslant \limsup_{r\to\infty} \sup_{z\in\mathbb{R}^m} \frac{\mu(Q_{[r]+1}(z))}{r^m} \\ &\leqslant \limsup_{r\to\infty} \frac{C([r]+1)^m}{r^m} = C.\end{aligned}$$

Since

$$\left|\mu\big(Q_r(z)\big) - \mu\big(Q_r(z-u)\big)\right| \leqslant \mu\big(Q_r(z) \setminus Q_r(z-u)\big) + \mu\big(Q_r(z-u) \setminus Q_r(z)\big),$$

in order the prove (b), it is clearly enough to show that

$$\lim_{r\to\infty} r^{-m} \mu\big(Q_r(z) \setminus Q_r(z-u)\big) = 0$$

uniformly for $z \in \mathbb{R}^m$ and $u \in Q_1(0)$. We have

$$Q_r(z) \setminus Q_r(z-u) = \bigcup_{j=1}^m Q_r(z) \cap \left\{x \in \mathbb{R}^m,\ |x_j + u_j - z_j| > \frac{r}{2}\right\} = \bigcup_{j=1}^m A_j$$

where, for $j = 1, \ldots, m$,

$$A_j = \left\{x \in \mathbb{R}^m, |x_j + u_j - z_j| > \frac{r}{2}, |x_i - z_i| \leqslant \frac{r}{2},\ i = 1, \ldots, m\right\}.$$

Note that we have the inclusion $A_j \subset B_j \cup C_j$, with B_j being the set

$$\left\{x \in \mathbb{R}^m,\ z_j - \frac{r}{2} \leqslant x_j \leqslant z_j - \frac{r}{2} + \frac{1}{2},\ |x_i - z_i| \leqslant \frac{[r]+1}{2},\ 1 \leqslant i \leqslant m,\ i \neq j\right\}$$

and C_j the set

$$\left\{x \in \mathbb{R}^m,\ z_j + \frac{r}{2} - \frac{1}{2} \leqslant x_j \leqslant z_j + \frac{r}{2}, |x_i - z_i| \leqslant \frac{[r]+1}{2},\ 1 \leqslant i \leqslant m,\ i \neq j\right\},$$

where $[r]$ denotes the integer part of r. It is clear that B_j is contained in the union of at most $([r]+1)^{m-1}$ cubes of side length 1 and the same is true for C_j. Hence, $\max\{\mu(B_j), \mu(C_j)\}$ is bounded by $C([r]+1)^{m-1}$. It follows immediately that

$$r^{-m} \mu\big(Q_r(z) \setminus Q_r(z-u)\big) \leqslant 2mC\big([r]+1\big)^{m-1} r^{-m} \to 0, \quad r \to \infty,$$

where the convergence is obviously uniform for $z \in \mathbb{R}^m$ and $u \in Q_1(0)$, which proves our claim. $\square$

Proposition 6.2. *Let μ be a positive translation-bounded measure on $\mathbb{R}^m$ such that, for some $\rho > 0$,*

$$\mathcal{F}\mu = \delta_0 \quad \text{on } B_\rho,$$

where B_ρ denotes the open ball centered at 0 with radius ρ in $\mathbb{R}^m$. Then, the Beurling density of μ, $D(\mu)$, exists and is equal to 1.

Proof. The statement that $D(\mu)$ exists and is equal to 1 is clearly equivalent to the fact that

$$\lim_{r\to\infty} r^{-m} \int\limits_{\mathbb{R}^m} \chi_Q\left(\frac{z-y}{r}\right) d\mu(y) = 1$$

uniformly for $z \in \mathbb{R}^m$, where $Q = Q_1(0)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^m)$ have the properties that $\varphi \geqslant 0$ on $\mathbb{R}^m$, that $\int_{\mathbb{R}^m} \varphi(x)\,dx = 1$ and that $\operatorname{supp}(\hat{\varphi}) \subset B_\rho$. Define, for $\eta > 0$, $\varphi_\eta(x) = \eta^{-m}\varphi(x/\eta)$, for $x \in \mathbb{R}^m$. Note that the function

$$\mathcal{F}\bigl\{r^{-m}(\chi_Q * \varphi_\eta)(\cdot/r)\bigr\}(\xi) = \hat{\chi}_Q(r\xi)\hat{\varphi}(\eta r\xi), \quad \xi \in \mathbb{R}^m,$$

has its support contained in B_ρ if $r > \eta^{-1}$ and, in that case, we have thus

$$\begin{aligned}\mathcal{F}\bigl\{\mu * r^{-m}(\chi_Q * \varphi_\eta)(\cdot/r)\bigr\}(\xi) &= \hat{\mu}(\xi)\hat{\chi}_Q(r\xi)\hat{\varphi}(\eta r\xi) = \delta_0(\xi)\hat{\chi}_Q(r\xi)\hat{\varphi}(\eta r\xi)\\ &= \hat{\chi}_Q(0)\hat{\varphi}(0)\delta_0(\xi) = \delta_0(\xi).\end{aligned}$$

It follows thus, by taking the inverse Fourier transform in the previous equality, that, if $r > \eta^{-1}$, we have

$$r^{-m} \int\limits_{\mathbb{R}^m} (\chi_Q * \varphi_\eta)\left(\frac{z-y}{r}\right) d\mu(y) = 1, \quad \text{for all } z \in \mathbb{R}^m.$$

Hence, for $r > \eta^{-1}$, we have

$$\begin{aligned}&r^{-m} \int\limits_{\mathbb{R}^m} \chi_Q\left(\frac{z-y}{r}\right) d\mu(y) - 1\\ &\quad = r^{-m} \int\limits_{\mathbb{R}^m} \chi_Q\left(\frac{z-y}{r}\right) - (\chi_Q * \varphi_\eta)\left(\frac{z-y}{r}\right) d\mu(y)\\ &\quad = r^{-m} \int\limits_{\mathbb{R}^m} \chi_Q\left(\frac{z-y}{r}\right) - \left(\int\limits_{\mathbb{R}^m} \chi_Q\left(\frac{z-y-u}{r}\right)\varphi_\eta(u)\,du\right) d\mu(y)\\ &\quad = r^{-m} \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} \left\{\chi_Q\left(\frac{z-y}{r}\right) - \chi_Q\left(\frac{z-y-u}{r}\right)\right\}\varphi_\eta(u)\,du\,d\mu(y).\end{aligned}$$

Using the Fubini–Tonelli theorem, we can write this last expression as

$$r^{-m}\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m}\chi_Q\left(\frac{z-y}{r}\right)-\chi_Q\left(\frac{z-y-u}{r}\right)d\mu(y)\right\}\varphi_\eta(u)\,du$$
$$=I_1(\eta,r,z)+I_2(\eta,r,z)$$

where we define, letting $B=\{x\in\mathbb{R}^m,\ |x|\leqslant 1\}$,

$$I_1(\eta,r,z)=r^{-m}\int_{\mathbb{R}^m\setminus B}\left\{\int_{\mathbb{R}^m}\chi_Q\left(\frac{z-y}{r}\right)-\chi_Q\left(\frac{z-y-u}{r}\right)d\mu(y)\right\}\varphi_\eta(u)\,du$$

and

$$I_2(\eta,r,z)=r^{-m}\int_{B}\left\{\int_{\mathbb{R}^m}\chi_Q\left(\frac{z-y}{r}\right)-\chi_Q\left(\frac{z-y-u}{r}\right)d\mu(y)\right\}\varphi_\eta(u)\,du.$$

By part (a) of Lemma 6.1, there exist $r_0>0$ and $C>0$ depending only on μ such that, if $r\geqslant r_0$

$$\left|I_1(\eta,r,z)\right|\leqslant 3C\int_{\mathbb{R}^m\setminus B}\varphi_\eta(u)\,du\to 0,\quad \text{as } \eta\to 0.$$

Thus, given $\epsilon>0$, we can choose $\eta>0$ small enough so that

$$3C\int_{\mathbb{R}^m\setminus B}\varphi_\eta(u)\,du<\frac{\epsilon}{2}.$$

For this fixed η, we have, using part (b) of Lemma 6.1, that

$$\left|I_2(\eta,r,z)\right|\leqslant\sup_{\substack{y\in\mathbb{R}^m\\ u\in Q}}r^{-m}\left|\int_{\mathbb{R}^m}\chi_Q\left(\frac{z-y}{r}\right)-\chi_Q\left(\frac{z-y-u}{r}\right)d\mu(y)\right|\to 0,$$

uniformly for $z\in\mathbb{R}^m$. It follows that, if r is large enough,

$$\left|I_1(\eta,r,z)\right|+\left|I_2(\eta,r,z)\right|<\epsilon,\quad \text{for all } z\in\mathbb{R}^m,$$

which proves our claim. □

An immediate consequence of the previous lemma and Theorem 4.4 is the following density result. A similar conclusion can be found in a paper by Kutyniok [16] in which the case of weighted irregular Gabor tight frames with a window in $L^2(\mathbb{R}^n)$ is considered.

Theorem 6.3. *Let μ be a translation-bounded, positive Borel measure on $\mathbb{R}^{2n}$ and let $g\in\mathcal{S}(\mathbb{R}^n)$. If there exists $h\in L^2(\mathbb{R}^n)$ such that $(h,g)_2=1$ and with h being a dual of Gabor type for the pair (g,μ), then necessarily $D(\mu)$ exists and is equal to* 1.

We note that the same result holds, again assuming that μ is translation-bounded, if we allow the dual h to be a distribution in $\mathcal{S}'(\mathbb{R}^n)$ as long as the duality is understood in the sense of the identity (4.6). We can use the previous result to obtain a generalization to weighted irregular Gabor systems of the so-called density theorem for regular Gabor systems in the particular case where the window is a Schwartz function and the generated system form a tight frame for $L^2(\mathbb{R}^n)$.

Theorem 6.4. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ be such that $\|g\|_2 = 1$ and consider the discrete irregular system*

$$\mathcal{G} := \left\{ w(x,\nu)^{1/2} e^{2\pi i \nu \cdot t} g(t-x) \right\}_{(x,\nu)\in\Lambda},$$

where Λ is a discrete set in the time–frequency space and w is a positive function defined on Λ. Let $D^-(\Lambda) = D^-(\rho)$, as defined above with ρ being the (unweighted) measure $\rho = \sum_{\lambda\in\Lambda} \delta_\lambda$. Then, if $\mathcal{G}$ forms a tight frame for $L^2(\mathbb{R}^n)$, we must have that $D^-(\Lambda) \geqslant 1$.

Proof. Multiplying the weight function w by an appropriate constant if necessary, we can assume that $\mathcal{G}$ forms a Parseval tight frame for $L^2(\mathbb{R}^n)$. Using the fact that $\|g\|_2 = 1$, it then follows by using the definition of a Parseval tight frame with $f(t) = e^{2\pi i t \cdot \nu_0} g(t - x_0)$ and $(x_0, \nu_0) \in \Lambda$, that

$$\|f\|_2^2 = \|g\|_2^2 = 1 = \sum_{(x,\nu)\in\Lambda} w(x,\nu) \left| \mathcal{V}_g f(x,\nu) \right|^2 \geqslant w(x_0,\nu_0)\|g\|_2^2 = w(x_0,\nu_0),$$

which shows that $w \leqslant 1$ on Λ. Defining the positive Borel measure μ as

$$\mu = \sum_{(x,\nu)\in\Lambda} w(x,\nu)\delta_{(x,\nu)},$$

we have thus that $\mathcal{F}\mu = \delta_{(0,0)}$ on a neighborhood of the origin by Theorem 3.2. Hence, using the previous lemma and the fact that $\rho \geqslant \mu$, we have,

$$D^-(\rho) \geqslant D^-(\mu) = 1,$$

which proves our claim. □

We conclude this paper by showing that there exist sets $\Lambda \subset \mathbb{R}^2$ which can be expressed as finite union of translates of a separable lattice and can have an arbitrarily large Beurling density while, at the same time, have the property that, for any window g in the Schwartz class, the system $\mathcal{G} := \{e^{2\pi i \nu t} g(t-x)\}_{(x,\nu)\in\Lambda}$ does not admit a dual of Gabor type in $L^2(\mathbb{R})$ (in the sense of Definition 4.1). This example illustrates the sharp contrast with known results [5,7] about irregular Gabor frames stating that if the window is sufficiently nice and the density is sufficiently high, the corresponding system does form a frame for $L^2(\mathbb{R})$. It emphasizes again the fundamental difference which exists between general irregular Gabor systems and those admitting a dual which is also of Gabor type.

Proposition 6.5. *Let $a > 1$ and choose $b_1, b_2 > 0$ such that $b_1/a \notin \mathbb{Q}$ and $b_2 \notin \mathbb{Q}$. Given positive integers K and L, define the set $\Lambda \subset \mathbb{R}^2$ by*

$$\Lambda = \left\{ (kb_1 + ma, lb_2 + n),\ 0 \leqslant k \leqslant K-1,\ 0 \leqslant l \leqslant L-1,\ m,n \in \mathbb{Z} \right\}.$$

Then, for any $g \in \mathcal{S}(\mathbb{R})$, the system $\mathcal{G} := \{e^{2\pi i\nu t} g(t-x)\}_{(x,\nu)\in\Lambda}$ does not admit a dual of Gabor type in $L^2(\mathbb{R})$.

Proof. Let $\Lambda_0 = a\mathbb{Z} \oplus \mathbb{Z}$. Consider the measures

$$\mu_0 = a \sum_{(x,\nu)\in\Lambda_0} \delta_{(x,\nu)} \quad \text{and} \quad \mu = \frac{a}{KL} \sum_{(x,\nu)\in\Lambda} \delta_{(x,\nu)}$$
$$= \frac{1}{KL} \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \delta_{(kb_1, lb_2)} * \mu_0.$$

Note that, since the density of the lattice Λ_0 is $a^{-1} < 1$, the system $\mathcal{G} := \{e^{2\pi i\nu t} g(t-x)\}_{(x,\nu)\in\Lambda_0}$ is not dense in $L^2(\mathbb{R})$ by the standard density theorem for regular Gabor systems (see [9]). In particular, there cannot exist a function $h \in L^2(\mathbb{R})$ satisfying the identity (4.11) with μ replaced by μ_0. On the other hand, if the system $\mathcal{G}$ admitted a dual $h \in L^2(\mathbb{R})$ with $(g,h)_2 = 1$, the identity (4.11) would hold for μ. Letting

$$P(\xi) = \frac{1}{K} \sum_{k=0}^{K-1} e^{-2\pi i \xi k b_1} \quad \text{and} \quad Q(\xi) = \frac{1}{L} \sum_{l=0}^{L-1} e^{-2\pi i \xi l b_2},$$

we have $\mathcal{F}\{\mu\}(\xi_1, \xi_2) = P(\xi_1) Q(\xi_2) \mathcal{F}\{\mu_0\}(\xi_1, \xi_2)$ and thus

$$\begin{aligned}
\mathcal{F}^S\{\mu\}(\xi_1, \xi_2) &= P(\xi_2) Q(-\xi_1) \mathcal{F}\{\mu_0\}(\xi_2, -\xi_1) \\
&= P(\xi_2) Q(-\xi_1) \sum_{(m,n)\in\mathbb{Z}^2} \delta_{(m, \frac{n}{a})}(\xi_1, \xi_2) \\
&= \sum_{(m,n)\in\mathbb{Z}^2} P\left(\frac{n}{a}\right) Q(-m) \delta_{(m, \frac{n}{a})}(\xi_1, \xi_2).
\end{aligned}$$

Using our hypothesis on b_1 and b_2, it follows that $P(\frac{n}{a}) Q(-m) \neq 0$ whenever $(m,n) \in \mathbb{Z}^2$. Thus, the support of $\mathcal{F}^S\{\mu\}$ is the same as that of $\mathcal{F}^S\{\mu_0\}$ and, since $P(0)Q(0) = 1$, we deduce that the identity (4.11) also holds if μ is replaced by μ_0, contradicting our earlier statement. Hence, $\mathcal{G}$ cannot admit a Gabor dual. □

Note that the Beurling density of the set Λ constructed in the previous theorem is KL/a and can thus be arbitrarily large.

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